COMPLEX SUBMANIFOLDS OF QUATERNIONIC KÄHLER MANIFOLDS *

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This is a survey on some basic results concerning complex and, in particular, Kähler submanifolds of a quaternionic Kähler manifold. Some problems which could be interesting to consider are outlined.

1. Preliminaries on quaternionic Kähler manifolds

We shall give a survey on (immersed) submanifolds having some interest to be considered into a quaternionic Kähler manifold. A **quaternionic Kähler manifold** will be denoted by $(\widetilde{M}^{4n}, \widetilde{g}, Q)$, where $\widetilde{g}$ is the Riemannian metric and $(\widetilde{g}, Q)$ is the quaternionic Hermitian structure on the $4n$-dimensional manifold $\widetilde{M} \equiv \widetilde{M}^{4n}$; the quaternionic structure Q, which is parallel with respect to the Levi-Civita connection $\widetilde{\nabla} = \nabla^{\widetilde{g}}$, is locally generated by an **admissible almost hypercomplex basis** $H = (J_1, J_2, J_3 = J_1 J_2)$ and the following identities hold:

$$\widetilde{\nabla}_X J_\alpha = \omega_\gamma(X) J_\beta - \omega_\beta(X) J_\gamma\,, \qquad X \in TM$$

where α, β, γ is a cyclic permutation of $1, 2, 3$ and the $\omega_\alpha, \alpha = 1, 2, 3$, are local 1-forms (depending on the choice of an admissible basis (J_α)). See for example [23],[1] for a basic introduction. Let us recall also that $(\widetilde{M}^{4n}, \widetilde{g})$ is an Einstein manifold and there is a decomposition of the curvature tensor

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of the form

$$\widetilde{R} = \nu R_{\mathbb{H}P^n} + \widetilde{W}$$

where $R_{\mathbb{H}P^n}$ is the curvature tensor of the quaternionic projective space $\mathbb{H}P^n$ with the standard metric, ν is a constant which is called the **reduced scalar curvature**, such that $K = 4n(n+2)\nu$ is the scalar curvature, and $\widetilde{W}$ is the **quaternionic Weyl tensor** which verifies the identity $[\widetilde{W}(X,Y), Q] = 0$ and has all contractions equal to zero.

For $n = 1$, *4-dimensional quaternionic Kähler manifolds are the same as Einstein anti-self-dual manifolds.*

The most basic 4n-dimensional quaternionic Kähler manifolds are respectively the *numerical quaternionic space* $\mathbb{H}^n$, $\nu = 0$, the *projective quaternionic space* $\mathbb{H}P^n$, $\nu > 0$, and its non compact dual, the *hyperbolic quaternionic space* $\mathcal{H}\mathbb{H}P^n$, $\nu < 0$, endowed with their canonical quaternionic Hermitian structure.

The last two models fall into the important classes of *Wolf spaces*, i.e. the compact quaternionic Kähler symmetric spaces, and of their non compact duals respectively .

2. Special submanifolds in quaternionic Kähler manifolds $(\widetilde{M}^{4n}, \widetilde{g}, Q)$

Submanifolds M of primary interest in $(\widetilde{M}^{4n}, \widetilde{g}, Q)$, where Q is a (rank-3) bundle of skew-symmetric endomorphisms, are the following.

Quaternionic submanifolds (M^{4m}, Q'): the tangent bundle of M is Q-invariant, $QTM = TM$, and $Q' = Q_{|TM}$.

By a classical result *they are totally geodesic*. Hence (M^{4m}, g, Q'), where $g = \widetilde{g}_{|TM}$, is a quaternionic Kähler manifold.

(Almost-) complex submanifolds (M^{2m}, J_1): the tangent bundle TM is J_1-invariant with respect to a section $J_1 \in \Gamma(Q_{|M}), J_1^2 = -\mathrm{Id}$. Then $(M^{2m}, g = \widetilde{g}_{|TM}, J = J_{1|TM})$ is an (almost-) Hermitian manifold.

As usual, the adverb *almost* is skipped if the almost complex structure $J = J_{1|TM}$ induced on M is integrable.

3. Complex submanifolds (M^{2m}, J_1)

Let (M^{2m}, J_1) be an almost complex submanifold of $(\widetilde{M}^{4n}, \widetilde{g}, Q)$. An **adapted basis** for (M^{2m}, J_1) is an admissible basis $H = (J_1, J_2, J_3)$ defined around a point of M^{2m}.

The problem of integrability of the almost complex structure $J = J_{1|TM}$, i.e. the vanishing of the Nijenhuis tensor $N(J)$, was studied in [2]. Let assume that (J_1, J_2, J_3) is an adapted basis and denote

$$\psi = (\omega_3 \circ J_1 - \omega_2)_{|TM} .$$

Then:

- *$N(J) = 0$ if and only if $\psi = 0$.*

Let consider the orthogonal decomposition

$$T_x M = \overline{T}_x M + \mathcal{D}_x , \quad x \in M,$$

where $\overline{T}_x M = T_x M \cap J_2 T_x M$ is the maximal quaternionic subspace of $T_x M$.

- *If $N(J)_x \neq 0$* then $\mathcal{D}_x = \mathrm{span}\{g^{-1}\psi, Jg^{-1}\psi\}$.
- *J is integrable if one of the following conditions holds:*
  $dim(\mathcal{D}_x) > 2$
  a) on an open dense set $U \subset M$

  or
  b) in a point x, if (M, J) is analytic.

Corollary 3.1 *If $dim(M) = 4k$ and $N(J) \neq 0$ on U dense in M then M^{4k} is a totally geodesic quaternionic submanifold.*

Corollary 3.2 *Let be $\nu > 0$.*
If (M^{4k}, g, J) is analytic and $g = \widetilde{g}_{|TM}$ is complete then (M^{4k}, g, J) is Hermitian.

Some problems:

- *Construction of examples of almost complex submanifolds which are not complex* $(m > 1)$. Also from the quoted results, it seems that such examples are rather rare.
- *Construction of pseudo-quaternionic submanifolds (non-holonomic quaternionic submanifolds $M^{4k+i}, i = 1, 2, 3,$ whose bundle QTM has minimal quaternionic rank $k+1$*; see [17], [6]). These submanifolds deserve some interest since they behave as submanifolds of a quaternionic submanifold $M^{4(k+1)}$ of minimal possible dimension. Of particular interest are the low dimensional cases M^2, M^3.

- *Evolution of almost complex surfaces M^2 following the mean curvature vector*. In [17] it was proved that such vector always belongs to the characteristic quaternionic line bundle QTM^2 (see also [3]); hence the evolution of M^2, eventually under appropriate hypothesis, could be useful to find a minimal surface as a limit (in this line of research, let see also [9]) or to generate a pseudo-quaternionic 3-submanifold M^3. In turn, since a pseudo-quaternionic M^3 has parallel characteristic quaternionic line bundle QTM^3, the evolution of M^3 following the mean curvature vector could produce a minimal 3-dimensional submanifold as a limit or generate a quaternionic submanifold M^4.

By referring (see [3]) to the twistor bundle of a quaternionic Kähler manifold,

$$\mathcal{Z} \xrightarrow{\pi} \widetilde{M}^{4n} \,,$$

where the *twistor space* $\mathcal{Z}$ is a complex contact manifold, it is natural to ask whether an almost complex submanifold (M^{2m}, J_1) of $\widetilde{M}^{4n}$ is **supercomplex**, i.e. it is the projection by π of a complex submanifold of $\mathcal{Z}$. In fact it was proved that, for $m > 1$, this happens exactly if and only if (M^{2m}, J_1) is complex. (The case of a complex surface (M^2, J_1) is a little more delicate to handle).

4. Kähler submanifolds

Let (M^{2m}, J_1) be an almost complex submanifold of $(\widetilde{M}^{4n}, \widetilde{g}, Q)$.

We note first that if $\nu \neq 0$ and $m \neq 3$ (which is still an open problem) the almost Hermitian submanifold $(M^{2m}, g = \widetilde{g}_{|TM}, J = J_{1|TM})$ is almost Kähler if and only if it is Kähler, see [2].

In [3] the following definition was proposed.

Definition 4.1 (M^{2m}, J_1) is Kähler if $\widetilde{\nabla}_X J_1 = 0 \,, \quad X \in TM$.

A standard example of Kähler submanifold of a quaternionic Kähler manifold is

$$\mathbb{C}P^n \hookrightarrow \mathbb{H}P^n \,.$$

It was also proved that

- a Kähler surface (M^2, J_1) is **superminimal**, i.e. supercomplex and minimal.

- if $m > 1$,

  (M^{2m}, J_1) is Kähler if and only if $(M^{2m}, g = \widetilde{g}_{|TM}, J = J_{1|TM})$ is a Kähler manifold;

- if $\nu \neq 0$,

  (M^{2m}, J_1) is Kähler if and only if (M^{2m}, J_1) is **totally complex**, i.e.

$$J_2 TM \perp TM\,.$$

Remark 4.1 A more general definition of "totally complex" submanifold (M^{2m}, J_1) could be considered by assuming only that: $QX \not\subseteq TM$, $\forall\, X \in TM$.

Kähler submanifolds (M, J_1) have interesting properties (see [13], [3]) being

- **minimal**,

in fact

- **pluriminimal** i.e. $\quad h(X,Y) + h(JX, JY) = 0\,, \quad \forall\, X, Y \in TM,$

where $h = 2^{\text{nd}}$ fundamental form. It follows from the stronger condition

$$h(JX, Y) - J_1 h(X,Y) = 0\,, \qquad \forall\, X, Y \in TM.$$

If $\nu \neq 0$, an (M^{2m}, J_1) pluriminimal and (super)complex is Kähler or quaternionic (h=0).

Problem: What can be said by using only "pluriminimality"?

In a quaternionic Kähler manifold with $\nu \neq 0$:

$$\dim(M^{2m}, J_1) \leq 2n \qquad .$$

Definition 4.2 If $\nu \neq 0$, $(M^{2n}, J_1) =$ **maximal** Kähler submanifold.

(McLean, F.E. Burstall call it a **complex-Lagrangian submanifold**, [16]).

In $(\widetilde{M}^{4n}, \widetilde{g}, Q)$, $\nu \neq 0$, *there are many maximal Kähler submanifolds*: it follows from a **generalization of Bryant construction for $\mathbb{C}P^3 \to S^4 \equiv \mathbb{H}P^1$ to the twistor bundle $\mathcal{Z} \xrightarrow{\pi} \widetilde{M}^{4n}$**. By the projection π, *there is a correspondence between* ***Legendrian submanifolds*** *of $\mathcal{Z}$ (i.e. maximal holomorphic horizontal submanifolds) and maximal Kähler submanifolds of $\widetilde{M}^{4n}$*, [3].

Totally geodesic maximal Kähler submanifolds of Wolf spaces were studied by M. Takeuchi, [25]. For classical Wolf spaces $\mathbb{H}P^n, G_2(\mathbb{C}^{n+2})$, $Gr_4^+(\mathbb{R}^{n+4})$ the situation is the following, where the inclusions have a natural geometrical content:

$$\mathbb{C}P^n \hookrightarrow \mathbb{H}P^n$$

$$\begin{array}{ccc} \mathbb{C}P^k \times \mathbb{C}P^{n-k} & \hookrightarrow & G_2(\mathbb{C}^{n+2}) \\ G_2^+(\mathbb{R}^{n+2}) & \hookrightarrow & G_2(\mathbb{C}^{n+2}) \end{array}$$

$$\begin{array}{ccc} G_2(\mathbb{C}^{n+2}) & \hookrightarrow & Gr_4^+(\mathbb{R}^{2n+4}) \\ Q_p(\mathbb{C}) \times Q_{n-p}(\mathbb{C})/\mathbb{Z}_2 & \hookrightarrow & Gr_4^+(\mathbb{R}^{n+4}) \end{array}$$

$(Q_p(\mathbb{C}) = \frac{SO(p+2)}{SO(p)\times SO(2)} =$ Complex hyperquadric).

In the following let assume $\nu \neq 0$.

A remarkable fact concerning a maximal Kähler submanifold $M \equiv M^{2n}$ is the identification:

$$J_2 : T^{\perp}M \xrightarrow{\sim} TM \,.$$

Then the Gauss-Codazzi equations can be expressed in terms of the tangent space TM, [2].

In particular, the second fundamental form h (locally) is identified with the **shape tensor** C on M defined by

$$C(X, Y, Z) = \langle J_2 h(X, Y), Z\rangle$$

which is symmetric with respect to X, Y, Z and satisfies the identities

$$C(JX, Y, Z) = C(X, JY, Z) = C(X, Y, JZ) \quad .$$

(Note: the shape tensor C is defined even if M is not maximal).[a]

5. Parallel Kähler submanifolds

If the Kähler submanifold (M^{2m}, J_1) is **parallel**, i.e. $\nabla' h = 0$, then the complex line bundle L generated by the shape tensor C (which is canonically defined and independent from the local section J_2) is a parallel line bundle.

[a] Added in proof: Recently K. Tsukada, [27], by basing on such Gauss-Codazzi equations and following a conjecture in [2], proved the fundamental theorem on the existence and uniqueness of isometric totally complex immersions for Kähler manifolds M^{2n} as submanifolds of $\mathbb{H}P^n$ and $\mathbb{H}H^n$

Let us first consider the case of *a maximal Kähler submanifold which is parallel, but not totally geodesic.*

Tsukada classified such submanifolds in $\mathbb{H}P^n$, [26].

Parallel maximal Kähler submanifolds of $\mathbb{H}P^n$

reducible

$$M^{2n} = \frac{SO_{n+1}}{SO_2 \cdot SO_{n-1}} \times \mathbb{C}P^1, (M^4 = \mathbb{C}P^1 \times \mathbb{C}P^1, M^6 = \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1)$$

$$M^8 = \tfrac{Sp_2}{U_2} \times \mathbb{C}P^1$$

irreducible

$$M^2 = \mathbb{C}P^1 \, , \, M^{12} = \frac{Sp_3}{U_3} \, , \, M^{18} = \frac{SU_6}{S(U_3 \times U_3)} \, , \, M^{30} = \frac{SO_{12}}{U_6} \, , \, M^{54} = \frac{E_7}{T \cdot E_6} \, .$$

The same classification holds for Kähler submanifolds which could be immersed as parallel maximal Kähler submanifolds into a quaternionic Kähler manifold with $\nu > 0$ and in the case of a quaternionic Kähler manifold with $\nu < 0$ the analogous classification result is obtained by considering as models the dual symmetric spaces of such M^{2n}, as one can prove by arguing as follows, [2].

If the Kähler submanifold (M^{2n}, J_1) is *parallel* then *the complex line bundle L generated by the shape tensor C is a parallel line bundle, cubic* (i.e. $L_x \subset S_3V^*, V =$ holomorphic tangent space $T^{1,0}M$), *of type* ν (i.e. the curvature form of the induced connection of L is proportional to the Kähler form of (M^{2n}, J_1) with coefficient of proportionality $i\nu$, $R^L = i\nu\Omega$). Then the classification of Kähler manifolds with parallel cubic line bundle reduces to the determination of the irreducible holonomy Lie algebras $\mathfrak{h}$ of Kähler manifolds such that the representation of $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$ in $V = T^{1,0}M$ has non trivial invariant quadratic or cubic form, i.e. $S^2(V^*)^{\mathfrak{h}'} \neq 0, S^3(V^*)^{\mathfrak{h}'} \neq 0$.

6. Parallel maximal Kähler submanifolds of Wolf spaces and their dual spaces

Definition 6.1 A submanifold $M \subset \widetilde{M}$ of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is called

- **curvature invariant** if $\widetilde{R}(T_xM, T_xM)T_xM \subset T_xM, \quad \forall x \in M$;
- **normal curvature invariant** if $\widetilde{R}(T_x^\perp M, T_x^\perp M)T_x^\perp M \subset T_x^\perp M, \forall x \in M$.

It is known that: (by Codazzi-Mainardi) *a parallel submanifold is curvature invariant* and (by Gauss) *a parallel submanifold of a locally symmetric space is itself a locally symmetric space.*

Definition 6.2 Let $\widetilde{M} = G/K$ be a homogeneous Riemannian manifold. Fix an orbit $\mathcal{V}$ of the isometry group G in the Grassmann bundle $\mathrm{Gr}_k(T\widetilde{M})$ of tangent k-planes of $\widetilde{M}$.
A k-dimensional submanifold $M \subset \widetilde{M}$ is called a *$\mathcal{V}$-submanifold if $T_xM \in \mathcal{V}$ for any $x \in M$. If $\mathcal{V}$ is (normal) curvature invariant, then any $\mathcal{V}$-submanifold is (normal) curvature invariant.*

([5]): *Any curvature invariant (in particular, any parallel) maximal Kähler submanifold (M^{2n}, J_1) of a quaternionic Kähler symmetric space $\widetilde{M}^{4n} \neq \mathbb{H}P^n$ is totally geodesic.* The proof bases on

- remark that the curvature identity

  $$\widetilde{g}(\widetilde{R}(J_2X, J_2Y)J_2Z, J_2T) = \widetilde{g}(\widetilde{R}(X,Y)Z,T)$$

  holds for any complex structure $J_2 \in Q$;
- results of H. Naitoh:
  - ([18]) *in a simply connected Riemannian symmetric space $\widetilde{M}$ a submanifold M is parallel and normal curvature invariant if and only if it is extrinsically symmetric*;
  - ([19]) *up to a short list of exceptions, a parallel normal curvature invariant, i.e. extrinsically symmetric, $\mathcal{V}$-submanifold of a symmetric space is in fact totally geodesic.*

Sketch proof. By the curvature identity above, M is also normal curvature invariant; hence, by [18], $\forall x \in M$ there exists an involutive isometry s_0 s.t. $(s_0)_{*|T_xM} = -\mathrm{Id}, (s_0)_{*|T_x^\perp M} = \mathrm{Id}$ and the totally geodesic submanifold $M(x) = \exp(T_xM)$ is an extrinsically symmetric maximal Kähler submanifold; then it follows that T_xM belongs to one of finitely many orbits $\mathcal{V} = G(V) \subset \mathrm{Gr}_{2n}T(G/K)$ and, by continuity reason, M is a $\mathcal{V}$-submanifold; by [19] one can conclude that M is totally geodesic if $\widetilde{M} \neq \mathbb{H}P^n$.

An elementary proof for $G_2(\mathbb{C}^{n+2})$ is also avalable, [4].

Theorem 6.1 *Let* (M^{2m}, J_1) *be a totally complex submanifold of a quaternionic Kähler manifold* $\widetilde{M}^{4n}$. *Assume that* M *is parallel. Then the first normal bundle* $N^1M \equiv h(TM, TM)$ *is totally complex, i.e.* $\langle h(X,Y), J_2 h(V,Z)\rangle = 0 \quad \forall X, YV, Z \in TM$. *Moreover, if* $\nu \neq 0$ *there are two cases:*

1) $C \equiv 0$, *i.e.* $N^1M \perp J_2TM$
2) $C \neq 0$, *and* M *is a locally symmetric Hermitian manifold with parallel cubic line bundle of type* ν.

The classification of parallel Kähler submanifolds in a quaternionic Kähler **symmetric** space reduces to the classification of parallel Kähler submanifolds in Hermitian or quaternionic Kähler symmetric spaces.

Theorem 6.2 *([5]) Let* (M^{2m}, J_1) *be a geodesically complete parallel Kähler submanifold of* $(\widetilde{M}^{4n}, \widetilde{g}, Q)$, $\nu \neq 0$, *and* $\overline{M}$ *the minimal totally geodesic submanifold of* $\widetilde{M}^{4n}$ *containing* M^{2m}.

1) *If* $C \equiv 0$ *then* $\overline{M}$ *is an Hermitian symmetric space and* (M^{2m}, J) *is a full parallel Kähler submanifold in* $\overline{M}$*;*
2) *If* $C \neq 0$, *and hence* (M^{2m}, J) *is a Kähler manifold with parallel cubic line bundle, then* $\overline{M}$ *is a quaternionic symmetric space of dimension* $4m$ *and* (M^{2m}, J) *is full in* $\overline{M}$.

An important class, also for physicists, of quaternionic Kähler manifolds which are homogeneous, but not necessarily symmetric, are the **Alekseevskian spaces**, [10].

Problem. *Classify parallel Kähler submanifolds in Alekseevskian quaternionic homogeneous spaces.*

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