# ON HOFER'S GEOMETRY OF THE SPACE OF LAGRANGIAN SUBMANIFOLDS *

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We survey some results in extending the Finsler geometry of the group of Hamiltonian diffeomorphisms of a symplectic manifold, known as Hofer's geometry, to the space of Lagrangian embeddings. Our intent is to illustrate some ideas of this still developing field, rather then to be complete or comprehensive.

## 1. Introduction

Let $(P,\omega)$ be a symplectic manifold and $H:[0,1]\times P\longrightarrow\mathbb{R}$ a smooth function. If $P$ is non-compact, we require $H$ to be constant outside a compact set. A diffeomorphism $\phi$ of $P$ is said to be *Hamiltonian* if $\phi=\phi_1^H$ where $\phi_t^H$ is a solution of

$$\frac{d}{dt}\,\phi_t^H(x)=X_H(\phi_t^H(x)),\quad \phi_0^H=\mathrm{Id}\,. \tag{1}$$

Here $X_H$ is Hamiltonian vector field, i.e. $\omega(X_H,\cdot)=dH(\cdot)$. Denote by $\mathrm{Ham}(P,\omega)$ the set of Hamiltonian diffeomorphisms of $P$. It is a Lie subgroup of the group of all diffeomorphisms of $P$. The Lie algebra $\mathrm{ham}(P,\omega)$ of $\mathrm{Ham}(P,\omega)$ is the algebra of all vector fields $\xi$ on $P$, tangent to some

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path of Hamiltonian diffeomorphisms $\psi_t$ that starts in Id. Such $\xi$ is also a Hamiltonian vector field. This important fact is not obvious and it was established by Banyaga [2]:

**Proposition 1.1** *[2] Every path $\psi_t$ of Hamiltonian diffeomorphisms in* $\mathrm{Ham}(P,\omega)$ *is also a Hamiltonian flow, i.e. there exist a smooth function* $F : [0,1] \times P \longrightarrow \mathbb{R}$ *such that*

$$\frac{d}{dt}\,\psi_t(x) = X_F(\psi_t(x)), \qquad \omega(X_F,\cdot) = dF\cdot.$$

Note that this function $F$ is determined by $\xi$ uniquely up to the constant. Vice versa, every smooth function $F$ defines a vector field from Lie algebra $\mathrm{ham}(P,\omega)$ and two functions $F$ and $F+c$ obviously define the same vector field. So we have the identification

$$\mathrm{ham}(P,\omega) \cong C^\infty(P)/\mathbb{R}. \tag{2}$$

Therefore, every norm $\|\cdot\|$ on $C^\infty(P)/\mathbb{R}$ defines the length of a path $\phi_t^H$ in $\mathrm{Ham}(P,\omega)$ in a natural way:

$$length(\phi_t^H) = \int_0^1 \left\| \frac{d\phi_t}{dt} \right\| dt = \int_0^1 \|H_t\| dt \tag{3}$$

which then naturally defines the (pseudo) distance in $\mathrm{Ham}(P,\omega)$: for two Hamiltonian diffeomorphisms $\phi_0$ and $\phi_1$ define

$$\delta(\phi_0,\phi_1) := \inf length(\phi_t), \tag{4}$$

where the infimum is taken over all Hamiltonian flows $\{\phi_t\}$ which connect $\phi_0$ and $\phi_1$. Different choice of norm on $\mathrm{Ham}(P,\omega)$ in (3) gives rise to different distance in (4). If we take $l^\infty$ norm on $\mathrm{Ham}(P,\omega)$:

$$\|H_t\|_\infty := \max_x H(t,x) - \min_x H(t,x),$$

the obtained length is called *Hofer length* and corresponding distance $\delta$ is called *Hofer distance*. It is easy to check that $\delta$ is a pseudo - distance on $\mathrm{Ham}(P,\omega)$, i.e. that it is positive, symmetric and satisfies the triangle inequality. Hofer length of paths is bi-invariant with respect to the action of the group Ham, which means that $length(\{\vartheta\phi_t\}) = length(\{\phi_t\vartheta\}) = length(\{\phi_t\})$, for all Hamiltonian diffeomorphisms $\vartheta$, so the Hofer distance also turns to be bi-invariant:

$$\delta(\vartheta\phi,\vartheta\psi) = \delta(\phi\vartheta,\psi\vartheta) = \delta(\phi,\psi).$$

This enables us to define the Hofer distance in the following way, equivalent to (4):

$$\delta(\phi,\psi) := \inf length(\varphi_t),$$

where the infimum is now taken over all Hamiltonian flows $\{\varphi_t\}$ which generate Hamiltonian diffeomorphism $\varphi = \phi\psi^{-1}$. The non - degeneracy of $\delta$ is a deep fact and it was proved by Hofer for the case $P = \mathbb{R}^{2n}$. The crucial part of the proof is inequality between symplectic capacity of a set $A$, $c(A)$ and the *displacement energy* of $A$, $e(A)$ which is defined as

$$e(A) := \inf\{\delta(\mathrm{Id},\phi) \mid \phi \in \mathrm{Ham}(P,\omega), A \cap \phi(A) = \emptyset\}. \tag{5}$$

**Theorem 1.1** *[8] For $A \subset \mathbb{R}^{2n}$ it holds: $c(A) \leq e(A)$.*

The proof of the non-degeneracy follows from Theorem 1.1. Indeed, if $\phi \neq \mathrm{Id}$ there exist $x \in \mathbb{R}^{2n}$ such that $\phi(x) \neq x$ and the open set $V \ni x$ such that $V \cap \phi(V) = \emptyset$. It is known that symplectic capacity $c(V)$ is strictly positive for an open set $V \subset \mathbb{R}^{2n}$, and it is obvious that $e(V) \leq \delta(\phi, \mathrm{Id})$. So we have that $\delta(\phi, \mathrm{Id})$ must be strictly positive. Polterovich [25] proved the non-degeneracy of Hofer distance for certain class of manifolds, and finally McDuff and Lalonde [11] proved it for all symplectic manifolds. Detailed and systematic treatment of the group of symplectomorphisms is given in the books by Hofer and Zehnder [9], McDuff and Salamon [14], Polterovich [27] and the survey article by McDuff [13].

## 2. Hofer's geometry for Lagrangian submanifolds

A submanifold $L$ of a symplectic manifold $(P,\omega)$ is called *Lagrangian* if $\dim L = \frac{1}{2}\dim P$ and $\omega|_{TL} = 0$. Lagrangian submanifolds are generalization of the group $\mathrm{Ham}(P,\omega)$ in the following sense: if $\phi$ is Hamiltonian diffeomorphism then the graph of $\phi$ is a Lagrangian submanifold of $(P \times P, \omega \oplus -\omega)$. Fix a closed Lagrangian submanifold $L_0$ of $P$. We will say that Lagrangian submanifold $L$ is Hamiltonian isotopic to $L_0$ if there exist $\psi \in \mathrm{Ham}(P,\omega)$ such that $L = \psi(L_0)$. Denote by $\mathcal{L} = \mathcal{L}(P,\omega,L_0)$ the space of all Lagrangian submanifolds that are Hamiltonian isotopic to $L_0$. Define a length of a path $\{L_t\}$, $L_t = \phi_t^H(L)$ in $\mathcal{L}$ as:

$$length(\{L_t\}) := \inf \int_0^1 (\max_x H(t,x) - \min_x H(t,x))dt, \tag{6}$$

where the infimum is taken over all $H$ such that $\phi_t^H(L_0) = L_t$. This definition is natural in the following sense. If we denote by $\mathcal{Y}$ the space of all Lagrangian submanifolds $L$ of $P$ that are diffeomorphic to $\Lambda$ then the tangent vector to $\mathcal{Y}$ at the point $L$ can be naturally identified with the space of closed one - forms on $L$:

$$T_L\mathcal{Y} = \{\eta \in \Omega^1(L) \mid d\eta = 0\}.$$

Indeed, if $j_t : \Lambda \longrightarrow P$ is a path in $\mathcal{Y}$, with $j_0(\Lambda) = L$, then the correspondence

$$X(p) := \left.\frac{d}{dt}\right|_{t=0} j_t(p) \quad \longleftrightarrow \quad \omega(\cdot, X(p)), \qquad p \in L$$

gives the claimed identification. With this identification, if $L_t$ is a path of Lagrangian submanifolds that are Hamiltonian isotopic to $L_0$, i.e. $L_t = \phi_t^H(L)$, then the tangent vector field to $L_t$ is the closed one - form $dH_t|_{L_t}$. Therefore, the length (6) of a path $L_t$ is defined as the integral of the norm of tangent vector field to $L_t$ (see [1] for more details). For $L_1, L_2 \in \mathcal{L}$ define a non - negative function $d$ as an infimum of lengths of all paths connecting $L_1$ and $L_2$:

$$d(L_1, L_2) := \inf\{\delta(\mathrm{Id}, \phi) \mid \phi \in \mathrm{Ham}(P, \omega),\ \phi(L_1) = L_2\}. \tag{7}$$

As in the case of Hofer's metric for Hamiltonian diffeomorphisms, one can easily check that $d$ is a pseudo-metric. It is also invariant under the action of the group $\mathrm{Ham}(P, \omega)$ on $\mathcal{L}$, given by $(\psi, L) \mapsto \psi(L)$. The non-trivial question is again whether $d$ is non-degenerate. In the case when $P$ is tame, which means geometrically bounded (the examples of tame manifolds are $\mathbb{R}^{2n}$, compact symplectic manifolds, cotangent bundles over compact manifolds), Chekanov proved the following:

**Theorem 2.1** *[4] Let $L_0$ be a closed embedded Lagrangian submanifold of a tame symplectic manifold $(P, \omega)$. Then the Hofer distance on $\mathcal{L}(P, \omega, L_0)$ is non-degenerate.*

The proof is based on the fact that every closed embedded Lagrangian submanifold has positive displacement energy (5). He also proved the following two theorems:

**Theorem 2.2** *[4] Let $d$ be a $\mathrm{Ham}(P, \omega)-$invariant metric on $\mathcal{L}(P, \omega, L_0)$. If $d$ is degenerate then it vanishes identically.*

**Theorem 2.3** *[4] Let $L_0$ be an embedded Lagrangian submanifold of a symplectic manifold $(P, \omega)$. If $\dim L_0 \geq 2$, then every $\mathrm{Ham}(P, \omega)-$invariant metric on $\mathcal{L}(P, \omega, L_0)$ is a multiply of $d$.*

The last claim fails for $\dim L_0 = 1$ and Chekanov [4] constructed a $\mathrm{Ham}(P,\omega)-$ invariant metric on $\mathcal{L}(P,\omega,\mathbb{S}^1)$ (where $P$ is a symplectic surface) which is not a multiply of $d$. If $L_0$ is manifold with boundary, then we have:

**Theorem 2.4** *[5] Let $L_0$ be a compact embedded submanifold of a tame symplectic manifold $(P,\omega)$. If the boundary of $L_0$ is not empty then any $\mathrm{Ham}(P,\omega)-$ invariant metric on $\mathcal{L}(P,\omega,L_0)$ is degenerate provided it is continuous in $C^1$ topology.*

If $H^1(L_0,\mathbb{R}) \neq 0$, then there are deformations of $L$ as Lagrangian embedding which are not Hamiltonian, denote by $\mathcal{L}^* = \mathcal{L}^*(P,\omega,L_0)$ the set of all such closed embedded Lagrangian deformations of $L_0$. The natural question is if there exists a $\mathrm{Ham}(P,\omega)-$ invariant metric on $\mathcal{L}^*$. The answer is not known in general case; Chekanov had two partial results, the first one gives the negative answer and the second one affirmative.

**Theorem 2.5** *[5] Let $L_1$,$L_2$ be closed embedded submanifolds (of positive dimension) of tame symplectic manifolds respectively $(P_1,\omega_1)$, $(P_2,\omega_2)$. If $L_1$ admits a closed 1-form without zeroes, then any $C^1-$continuous $\mathrm{Ham}(P_1\times P_2,\omega_1\otimes\omega_2)-$invariant metric on $\mathcal{L}^*(P_1\times P_2,\omega_1\otimes\omega_2,L_1\times L_2)$ is degenerate.*

**Theorem 2.6** *[5] If $L_0 = \mathbb{S}^1$ is an embedded (Lagrangian) submanifold of a symplectic surface $(P,\omega)$, then there exists a $C^\infty-$continuous $\mathrm{Ham}(P,\omega)-$invariant metric on $\mathcal{L}^*$.*

The existence of invariant metrics different from multiplies of Hofer's is still an open problem even in the case of the group $\mathrm{Ham}(P,\omega)$, i.e. the metric $\delta$. Eliashberg and Polterovich [6] showed that, if, in (3), one takes $l^p-$ norm instead of $l^\infty-$ norm, the induced metric on $\mathrm{Ham}(P,\omega)$ is degenerated. They used the fact that if $\delta$ in non - degenerated than the displacement energy (5) of every open set is strictly positive. There is an analogue of Theorem 2.2 for Hamiltonian diffeomorphisms, given also by Polterovich and Eliashberg:

**Theorem 2.7** *[6] If the bi-invariant pseudo-metric $\delta$ on $\mathrm{Ham}(M)$ for a closed symplectic manifold $M$ is degenerate, then it vanishes identically.*

## 3. Loops of Lagrangian submanifolds

Consider the loops of Hamiltonian diffeomorphisms, i.e. $\{\phi_t\} \in \mathrm{Ham}(P,\omega)$, such that $\phi_1 = \phi_0 = \mathrm{Id}$ and the classes of such loops which are Hamiltonian isotopic; the set of those classes is $\pi_1(\mathrm{Ham}(P,\omega))$. For a fixed $\gamma \in \pi_1(\mathrm{Ham}(P,\omega))$ define the norm of $\gamma$ by

$$\nu(\gamma) := \inf\{length(\{\phi_t^H\}) \mid H \in \mathrm{Ham}(P,\omega)\ [\phi_t] = \gamma\}.$$

The set $\{\nu(\gamma) \mid \gamma \in \pi_1(\mathrm{Ham}(p,\omega)\}$ is called *length spectrum.* Polterovich [27] used the length spectrum to estimate Hofer's distance $\delta$ from bellow. He also consider the positive and negative part of $\nu$:

$$\nu_+ := \inf_H \int_0^1 \max_x H(t,x)dt, \quad \nu_- := \inf_H \int_0^1 -\min_x H(t,x)dt$$

and calculated the exact value of $\nu_+$ in the case $P = \mathbb{S}^2$ (it turns out that the calculation of $\nu_+$ is non-trivial even in this case). Akveld and Salamon [1] considered the loops $L_t$ of exact Lagrangian manifolds (such that $L_t = \phi_t^H(L_0)$ for Hamiltonian isotopy $\phi_t$ of $P$ and fixed $L_0$). Analogously, they examined the minimal Hofer length in given Hamiltonian isotopy class:

$$\nu(L_t) := \inf\{length(\{\phi_t^H(L_t)\}) \mid H \in \mathrm{Ham}(P,\omega)\}$$

and calculated the exact values of $\nu$ for loops of projective Lagrangian planes. More precisely, let $\mathcal{L} = \mathcal{L}(\mathbb{C}P^n, \mathbb{R}P^n)$ a space of Lagrangian submanifolds that are diffeomorphic to $\mathbb{R}P^n$. Define the loop $\Lambda^k \subset \mathbb{S}^1 \times \mathbb{C}P^n$ as:

$$\Lambda^k := \bigcup_{t\in\mathbb{R}} \{e^{2\pi i t}\} \times \phi_{kt}(\mathbb{R}P^n)$$

where $\phi_t([z_0 : z_1 : \cdots : z_n]) := [e^{\pi i t} z_0 : z_1 : \cdots : z_n]$ and $k \in \mathbb{Z}$. Then it holds

**Theorem 3.1** *[1] Let $\omega \in \Omega^2(\mathbb{C}P^n)$ denote the Fubini - Study form that satisfies the normalization condition $\int_{\mathbb{C}P^n} \omega^n = 1$. Then*

$$\nu(\Lambda^k; \mathbb{C}P^n, \omega) = \frac{1}{2}$$

*for $k = 1, \ldots, n$ and $\nu(\Lambda^0) = 0$.*

This is a Lagrangian analogue of a theorem by Polterovich about loops of Hamiltonian diffeomorphisms of complex projective space. The proof uses the Gromov invariants of an associated symplectic fibration over the 2-disc with a Lagrangian subbundle over the boundary.

## 4. The relation between Hofer's metrics on the space of Hamiltonian diffeomorphisms and Lagrangian submanifolds

In a view of the generalization mentioned above, there is a canonical embedding

$$\mathrm{Ham}(P,\omega) \hookrightarrow \mathcal{L}(P\times P, \omega\oplus -\omega, \Delta), \quad \phi \mapsto \mathrm{graph}(\phi) \tag{8}$$

where $\Delta$ is the diagonal in $P \times P$. This embedding preserves Hofer's length of paths, so the natural question is whether it is isometric with respect to Hofer's distance, i.e. if $d(\mathrm{graph}(\mathrm{Id}), \mathrm{graph}(\phi)) = \delta(\mathrm{Id}, \phi)$ for every $\phi \in \mathrm{Ham}(P,\omega)$. Obviously $d \leq \delta$, because the infimum on the left is taken over a larger set. So the question is whether there exist some "tunnels" through $\mathcal{L}$ which do not pass trough the graphs of Hamiltonian diffeomorphisms and shorten the length between two graphs. The answer is positive, i.e. (8) is not isometric in general and it was given by Ostrover [24].

**Theorem 4.1** *[24] Let $P$ be a closed symplectic manifold with $\pi_2(P) = 0$ Then there exist a family $\phi_\tau$ in $\mathrm{Ham}(P,\omega)$ such that:*

- $\delta(\mathrm{Id}, \phi_\tau) \longrightarrow \infty$, *as* $\tau \longrightarrow \infty$,
- $d(\mathrm{graph}(\mathrm{Id}), \mathrm{graph}(\phi_\tau)) = c$ *for some positive constant* $c$.

This is a global result which says that the image of $\mathrm{Ham}(P,\omega)$ in $\mathcal{L}$ is distorted, unlike the situation described in [17] where it was proven that the space of Hamiltonian deformation of the zero section in the cotangent bundle is locally flat in Hofer metric. The corollary of Theorem 4.1 is that the group $\mathrm{Ham}(P,\omega)$ for $P$ closed with $\pi_2(P) = 0$ has an infinite diameter with respect to Hofer's metric. It is an interesting open question what is the diameter of $\mathcal{L}(P,\omega,L)$. In case of a two dimensional sphere $P = \mathbb{S}^2$ and the equator $L$ the space $\mathcal{L}(\mathbb{S}^2,\omega,L)$ is the space of simple closed curves that divide the sphere into two regions of equal area. Polterovich asked the question what is its diameter (see Problem 1.24 in [10]). Note that there is a result of Polterovich [26] that the diameter of $\mathrm{Ham}(\mathbb{S}^2,\omega)$ is infinite. In particular, it would be interesting to compare the diameters of $\mathrm{Ham}(M,\omega)$ and $\mathcal{L}(M\times M, \omega\oplus -\omega, \Delta)$.

## 5. The role of a Floer theory

Hofer's metric is basically a $C^0-$ property of Hamiltonian, while Hamiltonian diffeomorphisms are generated by its first derivative. Recall that classical Morse theory studies the relation between $C^0$ and $C^1$ objects, namely the level sets and the critical points of Morse functions. Floer theory, being a Morse theory for the action functional, is proved to be an useful tool for study the relation between $C^0$ and $C^1$ phenomena connected to Hamiltonian functions. Floer theory in Hofer geometry for Hamiltonian diffeomorphisms was used by Polterovich [27] (more precisely, he used the existence of certain perturbed pseudoholomorphic cylinders) to prove that every one - parameter subgroup of $\mathrm{Ham}(P,\omega)$ generated by a generic $H$ is locally minimal (i.e. its length is locally the Hofer's distance). Schwarz [28], Frauenfelder and Schlenk [7] and Oh (see, for example, [23] and the references there) also used Floer theory in Hofer geometry for Hamiltonian diffeomorphisms. Let us describe in a few details some applications of Floer homology for Lagrangian intersections to the Hofer's geometry for Lagrangian submanifolds. We refer the reader to [21, 22, 19, 20] and [16] for more details. Let $P = T^*M$ be a cotangent bundle over a compact smooth manifold $M$, $\omega$ a standard canonical form and $L_0 = O_M$ zero section. The proof of non-degeneracy of $d$ in this case given by Oh [21] is based on a study of invariants defined in the following way. Let $S \subset M$ be any compact submanifold. Define classical action functional $\mathcal{A}_H$ on a space

$$\Omega = \{\gamma : [0,1] \longrightarrow T^*M \mid \gamma(0) \in O_M,\ \gamma(1) \in N^*S\}$$

by

$$\mathcal{A}_H(\gamma) := \int_0^1 H(t,\gamma(t))dt - \int \gamma^*\theta.$$

The critical points of $\mathcal{A}_H$ are the solutions of

$$\begin{cases} \dot\gamma = X_H(\gamma) \\ \gamma(0) \in O_M,\ \gamma(1) \in N^*S. \end{cases} \tag{9}$$

It is possible to define Floer homology for $\mathcal{A}_H$ generated by paths $\gamma$ that satisfy (9), denote it by $HF_*(H,S)$ and the reduced Floer homology of chain complex filtered by action functional $\mathcal{A}_H$, denote it by $HF_*^{(-\infty,\lambda)}(H,S)$. Now define

$$\rho(H,S) := \inf\{\lambda \mid HF_*^{(-\infty,\lambda)}(H,S) \longrightarrow HF_*(H,S) \,\text{is surjective}\}. \tag{10}$$

Particularly, when $S$ consists of only one point, $S = \{x\}$, it holds

$$\max_x \rho(H,\{x\}) - \min_x \rho(H,\{x\}) \leq \int_0^1 (\max_x H(t,x) - \min_x H(t,x))dt. \tag{11}$$

After a certain normalization, it can be shown that the left - hand side in (11) depends only on $L := \phi_1^H(O_M)$, as well as that it is strictly positive, when $L \neq O_M$, so by taking an infimum over all $H$ that generate $L$ we obtain the non - degeneracy of Hofer's distance (see also [16]). Let us mention that there is more general construction of invariants $\rho$ given in [21], [22] and [20]. There they are parameterized by submanifolds $N \subset M$ and homological classes $a \in H_*(N)$. More precisely, the generators of Floer homology groups $HF^*(H, S : N)$ are the solutions of:

$$\begin{cases} \dot{\gamma} = X_H(\gamma) \\ \gamma(0) \in \operatorname{graph}(dS),\ \gamma(1) \in N^*N \end{cases}$$

where $S$ is a smooth function with some nice properties (see [21, 22, 20]). If

$$j_*^\lambda : HF_*^\lambda(H, S : N) \longrightarrow HF_*(H, S : N)$$

is the homomorphism induced by the natural inclusion, and $a \in H_*(N)$ define $\rho$ as:

$$\rho(a, H, S : N) := \inf\{\lambda \mid a \in \operatorname{Image}(j_*^\lambda F_*)\}$$

where $F_*$ is the mapping which establish the isomorphism between Floer and singular homology. The described invariants $\rho$ are the Floer-homology version on the invariants $c(L)$ constructed by Viterbo [30]. An approach to Viterbo's invariants via Morse homology is given in [15].

## 6. Geodesics

Let $P$ and $L$ be as in the previous section, so $\mathcal{L} = \mathcal{L}(T^*M, \omega, O_M)$ and $M$ is compact. By investigating the invariants $\rho$ (10) in a special case when $S = O_M$ (we will denote $\rho(H, O_M)$ by $\rho(H)$, or just $\rho$) one can obtain the description of geodesics in $\mathcal{L}$ with respect to Hofer's metric $d$ (7). This invariant $\rho$ is similar to the invariant $\gamma$ in [9] (see Proposition 12 and Proposition 13, pages 164-165) with similar properties. Let use describe the result in more details. The generators of Floer homology are the critical points of the action functional and in this case those are the solutions of:

$$\begin{cases} \dot{\gamma} = X_H(\gamma) \\ \gamma(0), \gamma(1) \in O_M \end{cases} \tag{12}$$

and the generators of $HF_*^{(-\infty,\lambda)}(H)$ are those paths $\gamma$ in (12) such that $\mathcal{A}_H(\gamma) \leq \lambda$. So we see that the number $\rho$ defined in (10) is a function $\rho : \operatorname{ham}(P, \omega) \longrightarrow \mathbb{R}$ (see (2)). We will assume here that the functions in ham$(P, \omega)$ are normalized in a certain way (see [17] for the details). This function $\rho$ has the following properties:

**Theorem 6.1**

1. *If* $L := \phi_1^H(O_M) = \phi_1^K(O_M)$, *then* $\rho(H) = \rho(K)$, *hence we can denote* $\rho(H)$ *by* $\rho(L)$.
2. $\rho(L) \in \mathrm{Spec}(H) := \{\mathcal{A}_H(\phi_t^H \circ (\phi_1^H)^{-1}(x)) \mid x \in O_M \cap L\}$
3. $$-\int_0^1 \max_{x \in T^*M}(H(t,x) - K(t,x))dt \leq \rho(H) - \rho(K) \leq$$
$$\leq -\int_0^1 \min_{x \in T^*M}(H(t,x) - K(t,x))dt;$$
*in particular,* $\rho$ *is monotone (if* $K \leq H$ *then* $\rho(K) \leq \rho(H)$*) and* $C^0-$ *continuous.*
4. $\rho(0) = 0$.
5. $\rho(\phi_1^H(O_M)) + \rho((\phi_1^H)^{-1}(O_M)) \leq d(O_M, \phi_1^H(O_M))$.
6. *If* $S : M \longrightarrow \mathbb{R}$ *is a smooth function, then* $\rho(-\pi^* S) = \max S$, *where* $\pi : T^*M \longrightarrow M$ *is a canonical projection.*

We refer the reader to [21, 17] for the proofs of Theorem 6.1. Denote by $\mathcal{F}(M)$ the space of the smooth function on $M$ normalized in the following way:

$$\mathcal{F}(M) := \{S \in C^\infty(M) \mid \int_M S(q)dq = 0\}$$

where $dq$ is the Lebesgue measure on $M$ induced by the Riemannian metric. Define a norm on $\mathcal{F}(M)$ by

$$\|S\| := \max_M S - \min_M S. \tag{13}$$

For any $C^1-$ small Hamiltonian deformation $L$ of $O_M$ (in the space of Lagrangian embeddings) there exist the unique smooth function $S$ on $M$ such that $L = \mathrm{graph}(dS)$. Denote this $C^1-$ neighborhood of $O_M$ by $\mathcal{G}$. Then we have the following result which is a consequence of the properties of invariant $\rho$ given above:

**Theorem 6.2** *(local flatness) [17] There exist* $C^1-$ *neighborhood* $\mathcal{U}$ *of* $0 \in \mathcal{F}(M)$ *such that the mapping:*

$$\Phi : \mathcal{U} \longrightarrow \mathcal{G}, \ L \mapsto \mathrm{graph}(dS)$$

*is an isometry with respect to the Hofer norm and the norm (13) on* $\mathcal{F}(M)$.

Compare this local result to Ostrover's global one, discussed in § 4. The corollary of Theorem 6.2 is the description of geodesics of Hofer distance in this particular case. The smooth path $L_t \in \mathcal{L}$ is *regular* if $\frac{d}{dt}L_t \neq 0$. We say that it is *minimal geodesic* if $length(\{L_t\}) = d(L_0, L_1)$ and *geodesic* if it is minimal geodesic locally on $[0,1]$. A Hamiltonian $H(t,x)$ is called *quasi-autonomous* if there exist $x_+, x_- \in \bigcup_t L_t$ such that

$$\max_x H(t,x) = H(t,x_+) \quad \text{and} \quad \min_x H(t,x) = H(t,x_-) \tag{14}$$

for every $t \in [0,1]$. Then it holds:

**Theorem 6.3** *[17] A regular path $\{L_t\}$ is a geodesic if and only if it is generated by a locally quasi-autonomous Hamiltonian function.*

Theorem 6.3 is a generalization of the analogous result for Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ by Bialy and Polterovich [3]. Lalonde and McDuff [12] obtained the description of geodesics in the group of Hamiltonian diffeomorphisms of general symplectic manifold. A different approach to study of geodesics, by means of second variation formula, was taken by Ustilovsky [29]. Bialy and Polterovich [3] proved that minimizing properties of geodesics are related to their *bifurcation diagram*. There is a similar result [18] for the case of Lagrangian submanifolds. Let us describe this result with the idea of proof. We call the path $L_t$ *strongly quasi autonomous* if it generated by Hamiltonian $H$ which is quasi autonomous on $\bigcup_t L_t$, and if there exists an open neighborhood $U$ of $\bigcup_{t\in[0,1]} L_t$ such that

$$H(t,x_-) \leq H(t,x) \leq H(t,x_+)$$

for $x \in U$ and $x_+, x_-$ as in (14). Set $H^s(t,x) := sH(st,x)$. Note that $H^s$ generates $\phi^H_{st}$. The *bifurcation diagram* corresponding to the Hamiltonian deformation $L_t = \phi^H_t(O_M)$ is the set

$$\Sigma(H) := \{(s,y) \in \mathbb{R}^2 \mid s \in (0,1],\, y \in \mathrm{Spec}(H^s)\}.$$

Let $H$ be strongly quasi autonomous and let

$$\begin{aligned}\gamma^H_+(s) &:= -\textstyle\int_0^s \min_{x\in\bigcup_{u\in[0,s]} L_u} H^s(t,x)dt,\\ \gamma^H_-(s) &:= -\textstyle\int_0^s \max_{x\in\bigcup_{u\in[0,s]} L_u} H^s(t,x)dt.\end{aligned}$$

Note that $\mathrm{graph}(\gamma^H_\pm) \subset \Sigma(H)$. A bifurcation diagram is called *simple* if the following two conditions are satisfied:

- Either $\gamma^H_+(s) = 0$ for all $s$, or for each $\tau > 0$ and for each continuous function $u : [\tau, 1] \longrightarrow \mathbb{R}$ such that $\mathrm{graph}(u) \subset \Sigma(H)$ and $u(\tau) = \gamma^H_+(\tau)$ holds $u(1) \geq \gamma^H_+(1)$.

- Either $\gamma_-^H(s) = 0$ for all $s$, or for each $\tau > 0$ and for each continuous function $u : [\tau, 1] \longrightarrow \mathbb{R}$ such that graph$(u) \subset \Sigma(H)$ and $u(\tau) = \gamma_-^H(\tau)$ holds $u(1) \leq \gamma_-^H(1)$.

Since for $L = \phi_1^H(O_M) = \phi_1^K(O_M)$ it holds

$$\mathcal{A}_H(\phi_t^H \circ (\phi_1^H)^{-1}(p)) - \mathcal{A}_K(\phi_t^K \circ (\phi_1^K)^{-1}(p)) = c(H, K)$$

for all $p \in L$, we see that the simplicity of a bifurcation diagram is a property of a Lagrangian deformation, independent of a particular choice of Hamiltonian generating it. We have the following:

**Theorem 6.4** *(Theorem 2 in [18]) Let $L[0,1] \longrightarrow \mathcal{L}$, $t \mapsto L_t$ be a strongly quasi autonomous path. Suppose that its bifurcation diagram is simple. Then length$(\{L_t\}) = d(L_0, L_1)$.*

The sketch of the proof is the following. It follows from the definition of simplicity and the properties of $\rho$ described in Theorem 6.1 that

$$\rho(H) \geq \gamma_+^H(1). \tag{15}$$

Recall that $(\phi_1^H)^{-1} = \phi_1^{\overline{H}}$ where $\overline{H} := -H(t, \phi_t^H(x))$, so we see that if $H$ is quasi autonomous then so is $\overline{H}$ (after cutting off $H$ away from $\bigcup L_t$, if necessary). Further, $\gamma_\pm^{\overline{H}}(t) = \gamma_\mp^H(t)$, so the bifurcation diagram $\Sigma(\overline{H})$ is also simple. Now we have

$$\begin{aligned} \rho((\phi_1^H)^{-1}(O_M)) = \rho(\phi_1^{\overline{H}}(O_M)) \geq -\textstyle\int_0^1 \min_x \overline{H}(t,x)dt = \\ -\textstyle\int_0^1 \min_x(-H(t,x))dt = \int_0^1 \max_x H(t,x)dt = -\gamma_-^H(1). \end{aligned} \tag{16}$$

Adding (15) and (16) we get

$$\gamma_+^H(1) - \gamma_-^H(1) \leq \rho(\phi_1^H(O_M)) + \rho((\phi_1^H)^{-1}(O_M)),$$

and using the property 5. in Theorem 6.1 we obtain:

$$\int_0^1 \|H\| dt = \gamma_+^H(1) - \gamma_-^H(1) \leq d(O_M, L_1)$$

and therefore

$$\int_0^1 \|H\| dt = length(\{L_t\}) = d(O_M, L_1).$$

In the above proof we used the fact that $\gamma_+^H(t) \neq 0$ for some $t$. In the case when $\gamma_+^H \equiv 0$ we need an auxiliary proposition which enables us to repeat the same proof for the path $\overline{L}_t := (\phi_t^H)^{-1}(O_M)$ instead of $L_t$. This is the content of Lemma 9 in [18]. (This Lemma is formulated somewhat imprecisely. It is obviously true in the case when $length(\{L_t\}) = \int_0^1 \|H\|dt$, which is not emphasized there, but that is the case in Theorem 6.4, i.e. Theorem 2 in [18].) There is one more criterion for minimality of geodesic which follows from Theorem 6.4. Strongly quasi autonomous path is called *admissible* if $x \in L_{t_0} \cap O_M$ for some $t_0 \in (0,1]$ implies $x \in L_t \cap O_M$ for every $t \in (0,1]$ and if $x_\pm$ are isolated, in a sense that $L_t \cap O_M \cap U_\pm = \{x_\pm\}$ for every $t$ and some open sets $U_\pm$. Then it holds:

**Theorem 6.5** *[18] Let $L : [0,1] \longrightarrow \mathcal{L}$, $t \mapsto L_t$ be an admissible path. Then*

$$length(\{L_t\}) = d(L_0, L_1).$$

The proof if based on applying Theorem 6.4 to the Hamiltonian $K$ which is chosen to be close to $H$ and such that its bifurcation diagram $\Sigma(K)$ turns out to be simple (where $H$ is the strongly quasi autonomous Hamiltonian which generates the path $L_t$).

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