MASSEY PRODUCTS IN GRADED LIE ALGEBRA COHOMOLOGY *

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We discuss Massey products in a $\mathbb{N}$-graded Lie algebra cohomology. One of the main examples is so-called "positive part" L_1 of the Witt algebra W. Buchstaber conjectured that $H^*(L_1)$ is generated with respect to non-trivial Massey products by $H^1(L_1)$. Feigin, Fuchs and Retakh represented $H^*(L_1)$ by trivial Massey products and the second part of the Buchstaber conjecture is still open. We consider the associated graded algebra $\mathfrak{m}_0$ of L_1 with respect to the filtration by its descending central series and prove that $H^*(\mathfrak{m}_0)$ is generated with respect to non-trivial Massey products by $H^1(\mathfrak{m}_0)$.

1. Introduction

In the last thirty years cohomological Massey products have found a lot of interesting applications in topology and geometry. The existence of non-trivial Massey products in $H^*(M, \mathbb{R})$ is an obstruction for a manifold M to be Kähler. A Kähler manifold M is formal [8], i.e. its real homotopy type is completely determined by the cohomology algebra $H^*(M, \mathbb{R})$. In their turn formal spaces have trivial Massey products.

An important feature of Massey products is the following: a blow-up of a symplectic manifold along its submanifold inherits non-trivial Massey products [3]. This idea was used by McDuff [20] in her construction of a simply connected symplectic manifold with no Kähler structure. The Massey products in cohomology of symplectic manifolds was the subject in [7]. Babenko and Taimanov considered an interesting family of symplectic

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nilmanifolds related to finite-dimensional quotients of a "positive part" L_1 of the Witt algebra W. Applying to them the symplectic blow-up procedure they constructed examples of simply connected non-formal symplectic manifolds in dimensions ≥ 10 [2]. A few time ago an 8-dimensional example was constructed by Fernandez and Muñoz [11] by means of another techniques.

The present article is devoted to the study of n-fold classical Massey products in the cohomology of $\mathbb{N}$-graded Lie algebras. We will focuss our attention to two main examples: L_1 and $\mathfrak{m}_0$. The infinite dimensional $\mathbb{N}$-graded Lie algebra L_1 is filtered by the ideals of the descending central series and one can consider its associated graded Lie algebra $\mathfrak{m}_0 = \mathrm{gr}_C L_1$. The reason of this interest comes from the relation of L_1 to the Landweber-Novikov algebra in the complex cobordisms theory discovered by Buchstaber and Shokurov [5].

It follows from the Goncharova theorem [14] that the cohomology algebra $H^*(L_1)$ has a trivial multiplication. Buchstaber conjectured that the algebra $H^*(L_1)$ is generated with respect to the Massey products by $H^1(L_1)$, moreover the corresponding Massey products can be chosen as non-trivial ones. The first part of Buchstaber's conjecture was proved by Feigin, Fuchs and Retakh [10]. However they represented Goncharova's basic cocycles of $H^*(L_1)$ using by trivial Massey products. Twelve years later Artel'nykh represented some of Goncharova's cocycles by non-trivial Massey products and unfortunately his brief article does not contain the proofs. Hence the original Buchstaber's conjecture is still open.

It was pointed out to the author by May that it follows from the Corollary 5.17 in [19] that the cohomology $H^*(L_1)$ is generated by $H^1(L_1)$ with respect to *matric* Massey products (generalized Massey products) and this property holds for some class of graded Lie algebras. The question of triviality or non triviality of corresponding matric Massey products have not been studied.

We recall the necessary information on the cohomology of graded Lie algebras in the first Section and study two main examples $H^*(L_1)$ [14] and $H^*(\mathfrak{m}_0)$ [12] in the Section 3. In the Section 4 we present May's approach to the definition of Massey products, his notion of formal connection developed by Babenko and Taimanov [3] for Lie algebras, we introduce also the notion of equivalent Massey products. The analogy with the classical Maurer–Cartan equation is especially transparent in the case of Massey products of 1-dimensional cohomology classes $\langle\omega_1, \dots, \omega_n\rangle$. The relation of this special case to the representations theory was discovered in [10, 9]. We discuss it in the Section 5.

The main result of the present article is the Theorem 6.4 stating that *the cohomology algebra $H^*(\mathfrak{m}_0)$ is generated with respect to the* ***non-trivial*** *Massey products by $H^1(\mathfrak{m}_0)$*. Another important result is the Theorem 6.3 that presents a list of equivalency classes of trivial Massey products of 1-cohomology classes from $H^1(\mathfrak{m}_0)$. We show that the Theorem 6.3 is related to Benoist's classification [4] of indecomposable finite-dimensional thread modules over $\mathfrak{m}_0$.

2. Cohomology of $\mathbb{N}$-graded Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ its linear representation (or in other words V is a $\mathfrak{g}$-module). We denote by $C^q(\mathfrak{g}, V)$ the space of q-linear skew-symmetric mappings of $\mathfrak{g}$ into V. Then one can consider an algebraic complex:

$$V \xrightarrow{d_0} C^1(\mathfrak{g}, V) \xrightarrow{d_1} C^2(\mathfrak{g}, V) \xrightarrow{d_2}$$

$$\dots \xrightarrow{d_{q-1}} C^q(\mathfrak{g}, V) \xrightarrow{d_q} \dots$$

where the differential d_q is defined by:

$$\begin{aligned}(d_q f)(X_1, \dots, X_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} \rho(X_i)(f(X_1, \dots, \hat{X}_i, \dots, X_{q+1}))+ \\ + \sum_{1 \le i < j \le q+1} (-1)^{i+j-1} f([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{q+1}).\end{aligned} \qquad (1)$$

The cohomology of the complex $(C^*(\mathfrak{g}, V), d)$ is called the cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in the representation $\rho : \mathfrak{g} \longrightarrow V$.

The cohomology of $(C^*(\mathfrak{g}, \mathbb{K}), d)$ ($V = \mathbb{K}$ and $\rho : \mathfrak{g} \longrightarrow \mathbb{K}$ is trivial) is called the cohomology with trivial coefficients of the Lie algebra $\mathfrak{g}$ and is denoted by $H^*(\mathfrak{g})$.

One can remark that $d_1 : C^1(\mathfrak{g}, \mathbb{K}) \longrightarrow C^2(\mathfrak{g}, \mathbb{K})$ of the $(C^*(\mathfrak{g}, \mathbb{K}), d)$ is the dual mapping to the Lie bracket $[\,,] : \Lambda^2 \mathfrak{g} \longrightarrow \mathfrak{g}$. Moreover the condition $d^2 = 0$ is equivalent to the Jacobi identity for $[,]$.

Definition 2.1 A Lie algebra $\mathfrak{g}$ is called $\mathbb{N}$-graded, if it is decomposed to a direct sum of subspaces such that

$$\mathfrak{g} = \oplus_i \mathfrak{g}_i, \ i \in \mathbb{N}, \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \ \forall\, i, j \in \mathbb{N}.$$

Example 2.1 The Lie algebra $\mathfrak{m}_0$ is defined by its infinite basis $e_1, e_2, \ldots, e_n, \ldots$ with commutator relations:

$$[e_1, e_i] = e_{i+1}, \ \forall \ i \geq 2.$$

The algebra $\mathfrak{m}_0$ has an infinite number of $\mathbb{N}$-gradings:

$$\mathfrak{m}_0 = \oplus_{i \in \mathbb{N}} \mathfrak{m}_{0i}, \ \mathfrak{m}_{01} = Span(e_1, e_2, \ldots, e_k), \ \mathfrak{m}_{0i} = Span(e_{i+k-1}), \ i \geq 2.$$

Remark 2.1 We always omit the trivial commutator relations $[e_i, e_j] = 0$ in the definitions of Lie algebras.

Example 2.2 Let us recall that the Witt algebra W is spanned by differential operators on the real line $\mathbb{R}^1$ with a fixed coordinate x

$$e_i = x^{i+1} \frac{d}{dx}, \ i \in \mathbb{Z}, \qquad [e_i, e_j] = (j-i) e_{i+j}, \ \forall \ i, j \in \mathbb{Z}.$$

We denote by L_1 the positive part of the Witt algebra, i.e. L_1 is a subalgebra of W spanned by all $e_i, \ i \geq 1$.

Obviously W is a $\mathbb{Z}$-graded Lie algebra with one-dimensional homogeneous components:

$$W = \oplus_{i \in \mathbb{Z}} W_i, \ W_i = Span(e_i).$$

Thus L_1 is a $\mathbb{N}$-graded Lie algebra.

Let $\mathfrak{g}$ be a Lie algebra. The ideals $C^k\mathfrak{g}$ of the descending central sequence determine a decreasing filtration C of the Lie algebra $\mathfrak{g}$

$$C^1\mathfrak{g} = \mathfrak{g} \supset C^2\mathfrak{g} \supset \cdots \supset C^k\mathfrak{g} \supset \ldots; \qquad [C^k\mathfrak{g}, C^l\mathfrak{g}] \subset C^{k+l}\mathfrak{g}.$$

One can consider the associated graded Lie algebra

$$\mathrm{gr}_C\mathfrak{g} = \bigoplus_{k \geq 1} (\mathrm{gr}_C\mathfrak{g})_k, \qquad (\mathrm{gr}_C\mathfrak{g})_k = C^k\mathfrak{g}/C^{k+1}\mathfrak{g}.$$

Proposition 2.1 *We have the following isomorphisms:*

$$\mathrm{gr}_C L_1 \cong \mathrm{gr}_C \mathfrak{m}_0 \cong \mathfrak{m}_0.$$

Remark 2.2 $\quad (\mathrm{gr}_C\mathfrak{m}_0)_1 = Span(e_1, e_2), \ (\mathrm{gr}_C\mathfrak{m}_0)_i = Span(e_{i+1}), \ i \geq 2.$

Let $\mathfrak{g} = \oplus_\alpha \mathfrak{g}_\alpha$ be a $\mathbb{Z}$-graded Lie algebra and $V = \oplus_\beta V_\beta$ is a $\mathbb{Z}$-graded $\mathfrak{g}$-module, i.e., $\mathfrak{g}_\alpha V_\beta \subset V_{\alpha+\beta}$. Then the complex $(C^*(\mathfrak{g}, V), d)$ can be equipped with the $\mathbb{Z}$-grading $C^q(\mathfrak{g}, V) = \bigoplus_\mu C^q_{(\mu)}(\mathfrak{g}, V)$, where a V-valued q-form c belongs to $C^q_{(\mu)}(\mathfrak{g}, V)$ if and only if for $X_1 \in \mathfrak{g}_{\alpha_1}, \dots, X_q \in \mathfrak{g}_{\alpha_q}$ we have

$$c(X_1, \dots, X_q) \in V_{\alpha_1+\alpha_2+\ldots+\alpha_q+\mu}.$$

This grading is compatible with the differential d and hence we have $\mathbb{Z}$-grading in the cohomology:

$$H^q(\mathfrak{g}, V) = \bigoplus_{\mu\in\mathbb{Z}} H^q_{(\mu)}(\mathfrak{g}, V).$$

Remark 2.3 The trivial $\mathfrak{g}$-module $\mathbb{K}$ has only one non-trivial homogeneous component $\mathbb{K} = \mathbb{K}_0$.

The exterior product in $\Lambda^*(\mathfrak{g})$ induces a structure of a bigraded algebra in the cohomology $H^*(\mathfrak{g})$:

$$H^q_k(\mathfrak{g}) \wedge H^p_l(\mathfrak{g}) \longrightarrow H^{q+p}_{k+l}(\mathfrak{g}).$$

Let $\mathfrak{g} = \oplus_\alpha \mathfrak{g}_\alpha$ be a $\mathbb{N}$-graded Lie algebra and $V = \oplus_\beta V_\beta$ is a $\mathbb{Z}$-graded $\mathfrak{g}$-module.

One can define a decreasing filtration $\mathcal{F}$ of $(C^*(\mathfrak{g}, V), d)$:

$$\mathcal{F}^0 C^*(\mathfrak{g}, V) \supset \dots \supset \mathcal{F}^q C^*(\mathfrak{g}, V) \supset \mathcal{F}^{q+1} C^*(\mathfrak{g}, V) \supset \dots$$

where the subspace $\mathcal{F}^q C^{p+q}(\mathfrak{g}, V)$ is spanned by $p+q$-forms c in $C^{p+q}(\mathfrak{g}, V)$ such that

$$c(X_1, \dots, X_{p+q}) \in \bigoplus_{\alpha \geq q} V_\alpha, \ \forall X_1, \dots, X_{p+q} \in \mathfrak{g}.$$

The filtration $\mathcal{F}$ is compatible with d.

Let us consider the corresponding spectral sequence $E_r^{p,q}$:

Proposition 2.2 *[13, 10]* $E_1^{p,q} = V_q \otimes H^{p+q}(\mathfrak{g})$.

Proof. We have the following natural isomorphisms:

$$\begin{gathered} C^{p+q}(\mathfrak{g}, V) = V \otimes \Lambda^{p+q}(\mathfrak{g}^*) \\ E_0^{p,q} = \mathcal{F}^q C^{p+q}(\mathfrak{g}, V)/\mathcal{F}^{q+1} C^{p+q}(\mathfrak{g}, V) = V_q \otimes \Lambda^{p+q}(\mathfrak{g}^*). \end{gathered} \tag{2}$$

Now the proof follows from the formula for the $d_0^{p,q} : E_0^{p,q} \longrightarrow E_0^{p+1,q}$:

$$d_0(v \otimes f) = v \otimes df,$$

where $v \in V, f \in \Lambda^{p+q}(\mathfrak{g}^*)$ and df is the standard differential of the cochain complex of $\mathfrak{g}$ with trivial coefficients.

3. Cohomology $H^*(L_1)$ and $H^*(\mathfrak{m}_0)$

Goncharova calculated in 1973 the cohomology $H^*(L_1)$.

Theorem 3.1 *[14] The Betti numbers* $\dim H^q(L_1) = 2$*, for every* $q \geq 1$*, more precisely*

$$\dim H^q_k(L_1) = \begin{cases} 1, & \text{if } k = \frac{3q^2 \pm q}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

We will denote in the sequel by $g^q_\pm$ the generators of the spaces $H^q_{\frac{3q^2\pm q}{2}}(L_1)$.

The numbers $\frac{3q^2\pm q}{2}$ are so called Euler pentagonal numbers. A sum of two arbitrary pentagonal numbers is not a pentagonal number, hence the algebra $H^*(L_1)$ has a trivial multiplication.

Example 3.1

1) $H^1(L_1)$ is generated by $g^1_- = [e^1]$ and $g^1_+ = [e^2]$;
2) the basis of $H^2(L_1)$ consists of two classes $g^2_- = [e^1 \wedge e^4]$ and $g^2_+ = [e^2 \wedge e^5 - 3e^3 \wedge e^4]$ of gradings 5 and 7 respectively.

The cohomology algebra $H^*(\mathfrak{m}_0)$ was calculated by Fialowski and Millionschikov [12].

There were introduced two operators in [12]:

1) $D_1 : \Lambda^*(e_2, e_3, \dots) \longrightarrow \Lambda^*(e_2, e_3, \dots)$,
$$D_1(e^2) = 0, \qquad D_1(e^i) = e^{i-1},\ \forall i \geq 3, \tag{3}$$
$$D_1(\xi \wedge \eta) = D_1(\xi) \wedge \eta + \xi \wedge D_1(\eta),\ \ \forall \xi, \eta \in \Lambda^*(e_2, e_3, \dots).$$
2) and its right inverse $D_{-1} : \Lambda^*(e^2, e^3, \dots) \longrightarrow \Lambda^*(e^2, e^3, \dots)$,
$$D_{-1}e^i = e^{i+1}, \qquad D_{-1}(\xi \wedge e^i) = \sum_{l \geq 0} (-1)^l D_1^l(\xi) \wedge e^{i+1+l}, \tag{4}$$
where $i \geq 2$ and ξ is an arbitrary form in $\Lambda^*(e^2, \dots, e^{i-1})$.

The sum in the definition (4) of D_{-1} is always finite because D_1^l decreases the second grading by l. For instance,

$$D_{-1}(e^i \wedge e^k) = \sum_{l=0}^{i-2} (-1)^l e^{i-l} \wedge e^{k+l+1}.$$

Proposition 3.1 *The operators D_1 and D_{-1} have the following properties:*

$$d\xi = e^1 \wedge D_1\xi, \quad e^1 \wedge \xi = dD_{-1}\xi, \quad D_1 D_{-1}\xi = \xi, \quad \xi \in \Lambda^*(e^2, e^3, \dots).$$

Theorem 3.2 *[12] The infinite dimensional bigraded cohomology $H^*(\mathfrak{m}_0) = \oplus_{k,q} H^q_k(\mathfrak{m}_0)$ is spanned by the cohomology classes of e^1, e^2 and of the following homogeneous cocycles:*

$$\omega(e^{i_1}\wedge\ldots\wedge e^{i_q}\wedge e^{i_q+1}) = \sum_{l\geq 0}(-1)^l D_1^l(e^{i_1}\wedge\cdots\wedge e^{i_q})\wedge e^{i_q+1+l}, \qquad (5)$$

where $q\geq 1,\ 2\leq i_1<i_2<\ldots<i_q$.

Formula (5) determines a homogeneous closed $(q{+}1)$-form of the second grading $i_1{+}\ldots{+}i_{q-1}{+}2i_q{+}1$. It has only one monomial in its decomposition of the form $\xi\wedge e^i\wedge e^{i+1}$ and it is $e^{i_1}\wedge\ldots\wedge e^{i_q}\wedge e^{i_q+1}$.

The whole number of linearly independent q-cocycles of the second grading $k{+}\frac{q(q+1)}{2}$ is equal to

$$\dim H^q_{k+\frac{q(q+1)}{2}}(\mathfrak{m}_0) = P_q(k) - P_q(k{-}1),$$

where $P_q(k)$ denotes the number of (unordered) partitions of a positive integer k into q parts.

Example 3.2

$$\begin{aligned}\omega(e^5\wedge e^6\wedge e^7) &= e^5\wedge e^6\wedge e^7 - e^4\wedge e^6\wedge e^8 + (e^3\wedge e^6 + e^4\wedge e^5)\wedge e^9\\ &- (e^2\wedge e^6 + 2e^3\wedge e^5)\wedge e^{10} + (3e^2\wedge e^5 + 2e^3\wedge e^4)\wedge e^{11}\\ &- 5e^2\wedge e^4\wedge e^{12} + 5e^2\wedge e^3\wedge e^{13}.\end{aligned}$$

The multiplicative structure in $H^*(\mathfrak{m}_0)$ was also found in [12] explicitly. In particular

$$[e^1]\wedge\omega(\xi\wedge e^i\wedge e^{i+1}) = 0,\ [e^2]\wedge\omega(\xi\wedge e^i\wedge e^{i+1}) = \omega(e^2\wedge\xi\wedge e^i\wedge e^{i+1}).$$

4. Massey products in cohomology.

In this section we follow [17] and [3] presenting the definitions of Massey products. Let $\mathcal{A} = \oplus_{l\geq 0}\mathcal{A}^l$ be a differential graded algebra over a field $\mathbb{K}$. It means that the following operations are defined: an associative multiplication

$$\wedge : \mathcal{A}^l \times \mathcal{A}^m \longrightarrow \mathcal{A}^{l+m},\ l,m\geq 0,\ l,n\in\mathbb{Z}.$$

such that $a\wedge b = (-1)^{lm}\, b\wedge a$ for $a\in\mathcal{A}^l$, $b\in\mathcal{A}^m$, and a differential d, $d^2=0$

$$d : \mathcal{A}^l \longrightarrow \mathcal{A}^{l+1},\quad l\geq 0,$$

satisfying the Leibniz rule $d\,(a \wedge b) = d\,a \wedge b + (-1)^l a \wedge d\,b$ for $a \in \mathcal{A}^l$.

Example 4.1 $\mathcal{A} = \Lambda^*(\mathfrak{g})$ is the cochain complex of a Lie algebra.

For a given differential graded algebra $(\mathcal{A}, d)$ we denote by $T_n(\mathcal{A})$ a space of all upper triangular $(n+1)\times(n+1)$-matrices with entries from $\mathcal{A}$, vanishing at the main diagonal. $T_n(\mathcal{A})$ has a structure of a differential algebra with a standard matrix multiplication, where matric entries are multiplying as elements of $\mathcal{A}$. A differential d on $T_n(\mathcal{A})$ is defined by

$$d\,A = (d\,a_{ij})_{1\le i,j\le n+1}. \tag{6}$$

An involution $a \to \bar{a} = (-1)^{k+1}a, a \in A^k$ of $\mathcal{A}$ can be extended to an involution of $T_n(\mathcal{A})$ as $\bar{A} = (\bar{a}_{ij})_{1\le i,j\le n+1}$. It satisfies the following properties:

$$\bar{\bar{A}} = A, \qquad \overline{AB} = -\bar{A}\bar{B}, \qquad \overline{d\,A} = -d\,\bar{A}.$$

Also we have the generalized Leibniz rule for the differential (6)

$$d\,(AB) = (d\,A)B - \bar{A}(d\,B).$$

The algebra $T_n(\mathcal{A})$ has a two-sided center $I_n(\mathcal{A})$ of matrices

$$\begin{pmatrix} 0 \dots 0\ \tau \\ 0 \dots 0\ 0 \\ \dots \\ 0 \dots 0\ 0 \end{pmatrix}, \quad \tau \in \mathcal{A}.$$

Definition 4.1 [3] A matrix $A \in T_n(\mathcal{A})$ is called the matrix of a formal connection if it satisfies the Maurer-Cartan equation

$$\mu(A) = d\,A - \bar{A}\cdot A \in I_n(\mathcal{A}). \tag{7}$$

Proposition 4.1 *[3] Let A be the matrix of a formal connection, then the entry $\tau \in \mathcal{A}$ of the matrix $\mu(A) \in I_n(\mathcal{A})$ in the definition (7) is closed.*

Proof. We have the following generalized Bianchi identity for the Maurer-Cartan operator $\mu(A) = d\,A - \bar{A}\cdot A$ (A is an arbitrary matrix):

$$d\,\mu(A) = \overline{\mu(A)}\cdot A + A\cdot\mu(A).$$

Indeed it's easy to verify the following equalities:

$$\begin{aligned} d\,\mu(A) = -d\,(\bar{A}\cdot A) &= -d\,\bar{A}\cdot A + A\cdot d\,A = \overline{d\,A}\cdot A + A\cdot d\,A \\ &= \overline{(\mu(A) + \bar{A}\cdot A)}\cdot A + A(\mu(A) + \bar{A}\cdot A) \\ &= \overline{\mu(A)}\cdot A - A\cdot\bar{A}\cdot A + A\cdot\mu A + A\cdot\bar{A}\cdot A \\ &= \overline{\mu(A)}\cdot A + A\cdot\mu(A). \end{aligned}$$

Now let A be the matrix of a formal connection, then the matrix $\mu(A)$ belongs to the center $I_n(\mathcal{A})$ and hence $d\mu(A) = 0$. One can think of $\mu(A)$ as the curvature matrix of a formal connection A.

Let A be an upper triangular matrix from $T_n(\mathcal{A})$. One can rewrite it in the following notation:

$$A = \begin{pmatrix} 0 & a(1,1) & a(1,2) & \dots & a(1,n-1) & a(1,n) \\ 0 & 0 & a(2,2) & \dots & a(2,n-1) & a(2,n) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a(n-1,n-1) & a(n-1,n) \\ 0 & 0 & 0 & \dots & 0 & a(n,n) \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Proposition 4.2 *A matrix $A \in T_n(\mathcal{A})$ is the matrix of a formal connection if and only if the following conditions on its entries hold on*

$$\begin{aligned} &a(i,i) = a_i \in \mathcal{A}^{p_i}, \quad i = 1, \dots, n; \\ &a(i,j) \in \mathcal{A}^{p(i,j)+1}, \qquad p(i,j) = \sum_{r=i}^{j} (p_r - 1); \\ &d\,a(i,j) = \sum_{r=i}^{j-1} \bar{a}(i,r) \cdot a(r+1,j), \quad (i,j) \neq (1,n). \end{aligned} \tag{8}$$

Proof. The system (8) is the Maurer-Cartan equation rewritten in terms of the entries of A and it coincides with the classical definition [15] of the defining system for a Massey product.

Definition 4.2 [15] A collection of elements, $A = (a(i,j))$, for $1 \leq i \leq j \leq n$ and $(i,j) \neq (1,n)$ is said to be a defining system for the product $\langle a_1, \dots, a_n \rangle$ if it satisfies (8).

In this situation the $(p(1,n)+2)$-dimensional cocycle

$$c(A) = \sum_{r=1}^{n-1} \bar{a}(1,r) a(r+1,n)$$

is called the related cocycle of the defining system A.

Remark 4.1 We saw that the notion of the defining system is equivalent to the notion of the formal connection. However one has to remark that an entry $a(1,n)$ of the matrix A of a formal connection does not belong to the corresponding defining system A, it can be taken as an arbitrary element from $\mathcal{A}$. In this event the only one possible nonzero entry τ of the Maurer-Cartan matrix $\mu(A)$ is equal to $-c(A) + da(1, n)$.

Definition 4.3 [15] The n-fold product $\langle a_1, \ldots, a_n \rangle$ is defined if there is at least one defining system for it (a formal connection A with entries $a_1, \ldots, a_n$ at the second diagonal). If it is defined, then $\langle a_1, \ldots, a_n \rangle$ consists of all cohomology classes $\alpha \in H^{p(1,n)+2}(\mathcal{A})$ for which there exists a defining system A (a formal connection A) such that $c(A)$ ($-\tau$ respectively) represents α.

Theorem 4.1 *[15, 3] The operation $\langle a_1, \ldots, a_n \rangle$ depends only on the cohomology classes of the elements $a_1, \ldots, a_n$.*

Proof. A changing of an arbitrary entry a_{ij}, $j > i$ of the matrix A of a formal connection to $a_{ij} + db$ leads to a replacement of A by

$$A' = A + db \cdot E_{ij} + A \cdot b \cdot E_{ij} - \bar{b} \cdot E_{ij} \cdot A,$$

where E_{ij} is a scalar matrix which has 1 on (i, j)-th place and zeroes on all others. For the corresponding Maurer-Cartan matrix we will have

$$\mu(A') = \mu(A) + d((A \cdot b \cdot E_{ij} - \bar{b} \cdot E_{ij} \cdot A) \cap I_n).$$

Definition 4.4 [15] A set of closed elements $a_i, i = 1, \ldots, n$ from $\mathcal{A}$ representing some cohomology classes $\alpha_i \in H^{p_i}(\mathcal{A}), i = 1, \ldots, n$ is said to be a defining system for the Massey n-fold product $\langle \alpha_1, \ldots, \alpha_n \rangle$ if it is one for $\langle a_1, \ldots, a_n \rangle$. The Massey n-fold product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is defined if $\langle a_1, \ldots, a_n \rangle$ is defined, in which case $\langle \alpha_1, \ldots, \alpha_n \rangle = \langle a_1, \ldots, a_n \rangle$ as subsets in $H^{p(1,n)+2}(\mathcal{A})$.

Example 4.2 For $n = 2$ the matrix A of a formal connection has a form $A = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$ and the matrix Maurer-Cartan equation is equivalent to two equations $da = 0$ and $db = 0$. Evidently $\langle \alpha, \beta \rangle = \bar{\alpha} \cdot \beta$.

Example 4.3 (Triple Massey products) Let α, β, and γ be the cohomology classes of closed elements $a \in \mathcal{A}^p$, $b \in \mathcal{A}^q$, and $c \in \mathcal{A}^r$. The Maurer-Cartan equation for

$$A = \begin{pmatrix} 0 & a & f & h \\ 0 & 0 & b & g \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

is equivalent to

$$d\,f = (-1)^{p+1}\, a \wedge b, \qquad d\,g = (-1)^{q+1}\, b \wedge c. \tag{9}$$

Hence the triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined if and only if

$$\alpha \cdot \beta = \beta \cdot \gamma = 0 \text{ in } H^*(\mathcal{A}).$$

If these conditions are satisfied then the Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined as a subset in $H^{p+q+r-1}(\mathcal{A})$ of the following form

$$\langle \alpha, \beta, \gamma \rangle = \left\{ [(-1)^{p+1} a \wedge g + (-1)^{p+q} f \wedge c] \right\}.$$

Since f and g are determined by (9) up to closed elements from $\mathcal{A}^{p+q-1}$ and $\mathcal{A}^{q+r-1}$ respectively, the triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is an affine subspace of $H^{p+q+r-1}(\mathcal{A})$ parallel to $\alpha \cdot H^{q+r-1}(\mathcal{A}) + H^{p+q-1}(\mathcal{A}) \cdot \gamma$.

Remark 4.2 We defined Massey products as the multi-valued operations in general. More often in the literature the triple Massey product is defined as a quotient-space $\langle \alpha, \beta, \gamma \rangle / (\alpha \cdot H^{q+r-1}(\mathcal{A}) + H^{p+q-1}(\mathcal{A}) \cdot \gamma)$ and it is single-valued in this case [13].

Definition 4.5 Let an n-fold Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ be defined. It is called trivial if it contains the trivial cohomology class: $0 \in \langle \alpha_1, \dots, \alpha_n \rangle$.

Proposition 4.3 *Let a Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ is defined. Then all Massey products $\langle \alpha_l, \dots, \alpha_q \rangle, 1 \le l < q \le n, q - l < n - 1$ are defined and trivial.*

Proof. It follows from (8).

Remark 4.3 The triviality of all Massey products $\langle \alpha_l, \dots, \alpha_q \rangle, 1 \le l < q \le n, q - l < n - 1$ is only a necessary condition for a Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ to be defined. It is sufficient only in the case $n = 3$.

Let us denote by $GT_n(\mathbb{K})$ a group of non-degenerate upper triangular $(n+1, n+1)$-matrices of the form:

$$C = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} & a_{1,n+1} \\ 0 & a_{2,2} & \dots & a_{2,n} & a_{2,n+1} \\ \dots & & \dots & & \dots \\ 0 & 0 & \dots & a_{n,n} & a_{n,n+1} \\ 0 & 0 & \dots & 0 & a_{n+1,n+1} \end{pmatrix}.$$

Proposition 4.4 *Let $A \in T_n(\mathcal{A})$ be the matrix of a formal connection and C an arbitrary matrix from $GT_n(\mathbb{K})$. Then the matrix $C^{-1}AC \in T_n(\mathcal{A})$ and satisfies the Maurer-Cartan equation, i.e. it is again the matrix of a formal connection.*

Proof.

$$d(C^{-1}AC) - \bar{C}^{-1}\bar{A}\bar{C} \wedge C^{-1}AC = C^{-1}\left(dA - \bar{A} \wedge A\right) C = 0.$$

Example 4.4 Let $A \in T_n(\mathcal{A})$ be the matrix of a formal connection (defining system) for a Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$. Then a matrix $C^{-1}AC$ with

$$C = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & x_1 & \dots & 0 & 0 \\ \dots & & \dots & & \dots \\ 0 & 0 & \dots & x_1 \dots x_{n-1} & 0 \\ 0 & 0 & \dots & 0 & x_1 \dots x_{n-1} x_n \end{pmatrix}$$

is a defining system for $\langle x_1\alpha_1, \dots, x_n\alpha_n \rangle = x_1 \dots x_n \langle \alpha_1, \dots, \alpha_n \rangle$.

Definition 4.6 Two matrices A and A' of formal connections from $T_n(\mathcal{A})$ are equivalent if there exists a matrix $C \in GL(n{+}1, \mathbb{K})$ such that

$$A' = C^{-1}AC.$$

Example 4.5 Triple products $\langle \alpha, \beta, \gamma \rangle$ and $\langle x\alpha, y\beta, z\gamma \rangle$, where $x, y, z \neq 0$, are equivalent with

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & xy & 0 \\ 0 & 0 & 0 & xyz \end{pmatrix}$$

and

$$\langle x\alpha, y\beta, z\gamma \rangle = xyz \langle \alpha, \beta, \gamma \rangle, \quad x, y, z \in \mathbb{K}.$$

Remark 4.4 Following the original Massey work [16] some higher order cohomological operations that we call now Massey products were introduced in the 60s in [15] and [17]. The relation between Massey products and the Maurer–Cartan equation was first noticed by May [17] and this analogy was not developed until [3].

In the present article we deal only with Massey products of non-trivial cohomology classes. It is possible to take some of them trivial, but in this situation is more natural to work with so–called matric Massey products that were first introduced by May [17].

5. Massey products and thread modules

Let $T_n(\mathbb{K})$ be a Lie algebra of upper triangular $(n+1, n+1)$-matrices over a field $\mathbb{K}$ of zero characteristic and $\rho : \mathfrak{g} \longrightarrow T_n(\mathbb{K})$ be a representation of a Lie algebra $\mathfrak{g}$.

Example 5.1 We take $n = 1$ and consider a linear map

$$\rho : x \in \mathfrak{g} \longrightarrow \begin{pmatrix} 0 & a(x) \\ 0 & 0 \end{pmatrix}.$$

It is evident that ρ is a Lie algebra homomorphism if and only if the linear form $a \in \mathfrak{g}^*$ is closed

$$da(x, y) = a([x, y]) = a(x)a(y) - a(y)a(x) = 0, \ \forall x, y \in \mathfrak{g}.$$

In other words the matrix $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ satisfies the "strong" Maurer-Cartan equation $dA - \bar{A} \wedge A = 0$.

Remark 5.1 We recall that we defined in the Section 4 the involution of a graded algebra $\mathcal{A}$ as $\bar{a} = (-1)^{k+1}a, a \in \mathcal{A}^k$. Thus for a matrix A with entries from $\mathfrak{g}^*$ we have $\bar{A} = A$. One has to remark that $\bar{a}$ differs by the sign from the definition of $\bar{a}$ in [15], however in [18] one meets our sign rule.

Proposition 5.1 *A matrix A with entries from $\mathfrak{g}^*$ defines a representation $\rho : \mathfrak{g} \longrightarrow T_n(\mathbb{K})$ if and only if A satisfies the strong Maurer-Cartan equation*

$$dA - \bar{A} \wedge A = 0.$$

Proof. $$(dA - \bar{A} \wedge A)(x, y) = A([x, y]) - [A(x), A(y)], \ \forall x, y \in \mathfrak{g}.$$

Example 5.2 For $n = 2$ the matrix A of a representation ρ has a form $A = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$, where $a, b, c \in \mathfrak{g}^*$ and the strong Maurer-Cartan equation is equivalent to the following equations on entries a, b, c:

$$da = 0, \qquad db = 0, \qquad dc = a \wedge b.$$

The Lie algebra $T_n(\mathbb{K})$ has a one-dimensional center $I_n(\mathbb{K})$ spanned by the matrix

$$\begin{pmatrix} 0 \dots 0 \, 1 \\ 0 \dots 0 \, 0 \\ \dots \\ 0 \dots 0 \, 0 \end{pmatrix}.$$

One can consider an one-dimensional central extension

$$0 \longrightarrow \mathbb{K} \cong I_n(\mathbb{K}) \longrightarrow T_n(\mathbb{K}) \xrightarrow{\pi} \tilde{T}_n(\mathbb{K}) \longrightarrow 0.$$

Proposition 5.2 *[10, 9] Fixing a Lie algebra homomorphism $\tilde{\varphi} : \mathfrak{g} \longrightarrow \tilde{T}_n(\mathbb{K})$ is equivalent to fixing a defining system A with elements from $\mathfrak{g}^* = \Lambda^1(\mathfrak{g})$. The related cocycle $c(A)$ is cohomologious to zero if and only if $\tilde{\varphi}$ can be lifted to a homomorphism $\varphi : \mathfrak{g} \longrightarrow T_n(\mathbb{K})$, $\tilde{\varphi} = \pi\varphi$.*

Taking a $(n+1)$-dimensional linear space V over $\mathbb{K}$ and a representation $\varphi : \mathfrak{g} \longrightarrow T_n(\mathbb{K})$ one gets a $\mathfrak{g}$-module structure of V defined in the coordinates $x = (x_1, \dots, x_{n+1})$ with respect to some fixed basis $v_1, \dots, v_{n+1}$ of V

$$gv = \varphi(g)x, \quad g \in \mathfrak{g}, \ v \in V.$$

Definition 5.1 Two representations $\varphi : \mathfrak{g} \longrightarrow T_n(\mathbb{K})$ and $\varphi' : \mathfrak{g} \longrightarrow T_n(\mathbb{K})$ are called isomorphic (equivalent) if there exist two bases $v_1, \dots, v_{n+1}$ and $v'_1, \dots, v'_{n+1}$ in a $(n+1)$-dimensional linear space V such that the corresponding $\mathfrak{g}$-module structures coincide $\rho(g)v = \rho'(g)v$. Or in other words, if there exists a matrix $C \in GL(n+1, \mathbb{K})$ such that

$$\varphi'(g) = C^{-1}\varphi(g)C, \ \forall \, g \in \mathfrak{g}.$$

It is evident that this definition is equivalent to the Definition 4.6 when we consider a Massey product $\langle \omega_1, \dots, \omega_n \rangle$ of 1-cohomology classes $\omega_1, \dots, \omega_n$.

Proposition 5.3 $\qquad [\mathrm{gr}_C \mathfrak{g}, \mathrm{gr}_C \mathfrak{g}] = \oplus_{i \geq 2} (\mathrm{gr}_C \mathfrak{g})_i.$

Proof. An inclusion $[\mathrm{gr}_C\mathfrak{g}, \mathrm{gr}_C\mathfrak{g}] \subset \oplus_{i\geq 2}(\mathrm{gr}_C\mathfrak{g})_i$ is evident for the graded Lie algebra $\mathrm{gr}_C\mathfrak{g} = \oplus_{i\geq 1}(\mathrm{gr}_C\mathfrak{g})_i$. Hence it is sufficient to prove an inclusion $(\mathrm{gr}_C\mathfrak{g})_i \subset [\mathrm{gr}_C\mathfrak{g}, \mathrm{gr}_C\mathfrak{g}]$ for an arbitrary $i \geq 2$. The last one in its turn follows from

$$(\mathrm{gr}_C\mathfrak{g})_i = [\mathfrak{g}, C^{i-1}\mathfrak{g}] + C^{i+1}\mathfrak{g} = [\mathfrak{g}+C^2\mathfrak{g}, C^{i-1}\mathfrak{g}+C^i\mathfrak{g}].$$

Corollary 5.1 *We have isomorphisms*

$$H^1(\mathfrak{g}) \cong (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^* \cong (\mathrm{gr}_C\mathfrak{g})_1^* \cong H^1(\mathrm{gr}_C\mathfrak{g}).$$

Definition 5.2 A homomorphism $\rho : \mathfrak{g} \longrightarrow \mathfrak{h}$ of two $\mathbb{N}$-graded Lie algebras $\mathfrak{g} = \oplus_i\mathfrak{g}_i$ and $\mathfrak{h} = \oplus_i\mathfrak{h}_i$ is called graded if

$$\rho(\mathfrak{g}_i) \subset \mathfrak{h}_i, \quad \forall i \in \mathbb{N}.$$

Definition 5.3 Let $\rho : \mathfrak{g} \longrightarrow T_n(\mathbb{K})$ be a representation. We call a representation of graded Lie algebras $\tilde{\rho} : \mathrm{gr}_C\mathfrak{g} \longrightarrow \mathrm{gr}_C T_n(\mathbb{K})$, defined by the rule

$$\tilde{\rho}(x + C^{k+1}\mathfrak{g}) = \rho(x) + C^{k+1}T_n(\mathbb{K}), \quad x + C^{k+1}\mathfrak{g} \in C^k\mathfrak{g}/C^{k+1}\mathfrak{g},\ k \geq 1,$$

an associated graded representation to ρ.

Remark 5.2 Let $\tilde{\rho} : \mathfrak{g} \longrightarrow T_n(\mathbb{K})$ be some representation of a Lie algebra $\mathfrak{g}$ such as $\mathfrak{g} \cong \mathrm{gr}_C\mathfrak{g}$. It is not hard to describe the corresponding matrix A of a formal connection:

1) the first diagonal is of zeroes (like all matrices from $T_n(\mathbb{K})$);
2) the second one contains only elements from $\mathfrak{g}^*$ of degree one;
3) the k-th diagonal consists only of elements of degree $k-1$.

Example 5.3 We take $n = 3$ and a homomorphism $\rho : \mathfrak{m}_0 \longrightarrow T_3(\mathbb{K})$ is defined by

$$\rho(e_1) = \begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix},\ \rho(e_2) = \begin{pmatrix} 0&1&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}.$$

On can to define ρ only on e_1 and e_2 because the algebra $\mathfrak{m}_0$ is generated by these two elements.

The corresponding matrix A of a formal connection is equal to

$$A = \begin{pmatrix} 0 & e^2 & -e^3 & e^4 + e^1 \\ 0 & 0 & e^1 & e^2 \\ 0 & 0 & 0 & e^1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the associated graded representation $\tilde{\rho} : \mathfrak{m}_0 \longrightarrow T_3(\mathbb{K})$ we have

$$\tilde{A} = \begin{pmatrix} 0 & e^2 & -e^3 & e^4 \\ 0 & 0 & e^1 & 0 \\ 0 & 0 & 0 & e^1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We recall that elements e^1 and e^2 have grading one, e^3 and e^4 have gradings 2 and 3 in $\Lambda^*(\mathfrak{m}_0) = \Lambda^*(\mathrm{gr}_C\mathfrak{m}_0)$ respectively.

Identifying the spaces $H^1(\mathfrak{g})$ and $H^1(\mathrm{gr}_C\mathfrak{g})$ we come to the following proposition:

Proposition 5.4 *Let a Massey product $\langle\omega_1, \omega_2, \ldots, \omega_n\rangle$ be defined and trivial in $H^2(\mathfrak{g})$ for some 1-cohomology classes $\omega_i \in H^1(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. Then $\langle\omega_1, \omega_2, \ldots, \omega_n\rangle$ is also defined and trivial in $H^2(\mathrm{gr}_C\mathfrak{g})$.*

Proposition 5.5 *Let $\mathfrak{g}$ be a Lie algebra such that $\mathfrak{g} \cong \mathrm{gr}_C\mathfrak{g}$ and a Massey product $\langle\omega_1, \omega_2, \ldots, \omega_n\rangle$ be defined and trivial for some $\omega_i \in H^1(\mathfrak{g})$. Then there exists a graded defining system A (the matrix A of a graded formal connection) for $\langle\omega_1, \omega_2, \ldots, \omega_n\rangle$.*

Definition 5.4 A thread module over a $\mathbb{N}$-graded Lie algebra $\mathfrak{g} = \oplus_i\mathfrak{g}_i$ is a $\mathbb{N}$-graded $\mathfrak{g}$-module $V = \oplus_{i\in\mathbb{N}}V_i$ such as

$$\dim V_i = 1, \quad \mathfrak{g}_i V_j \subset V_{i+j}, \ \forall i, j \in \mathbb{N}.$$

Fixing a basis $\{f_j\}$, $j = 1, \ldots, n+1$, in a $(n+1)$-dimensional thread module $V = \oplus_{j=1}^{n+1}V_j$, such that $f_j \in V_j$, gives us a representation of $\mathfrak{g}$ by lower triangular matrices. Taking the dual module V^*

$$\left(\rho^*(g)\tilde{f}\right)(v) = \tilde{f}(\rho(g)v)$$

with a basis $f^j \in V_j^*$, $j = 1, \ldots, n+1$ we will get a representation $\rho^* : \mathfrak{g} \longrightarrow T_n(\mathbb{K})$ by upper triangular matrices.

Or one can change the ordering of the basis of V considering a new basis $v'_1 = v_{n+1}, \ldots, v'_{n+1} = v_1$.

Proposition 5.6 *Let a Massey product $\langle \omega_1, \omega_2, \ldots, \omega_n \rangle$ be defined and trivial in $H^2(\mathfrak{g})$ for some 1-cohomology classes $\omega_i \in H^1(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. Then $\langle x_1\omega_1, x_2\omega_2, \ldots, x_n\omega_n \rangle$ is also defined and trivial for any choice of non-zero constants $x_1, x_2, \ldots, x_n$.*

6. Massey products in $H^*(L_1)$ and $H^*(\mathfrak{m}_0)$

We recall that the algebra $H^*(L_1)$ has a trivial multiplication. Buchstaber conjectured that the algebra $H^*(L_1)$ is generated with respect to the Massey products by $H^1(L_1)$, moreover all corresponding Massey products can be chosen non-trivial. The first part of Buchstaber's conjecture was proved by Feigin, Fuchs and Retakh [10].

Theorem 6.1 *[10] For any $q \geq 2$*

$$g_-^q \in \mathbb{K}\langle g_+^{q-1}, \underbrace{e^1, \ldots, e^1}_{2q-1} \rangle, \qquad g_+^q \in \mathbb{K}\langle g_+^{q-1}, \underbrace{e^1, \ldots, e^1}_{3q-1} \rangle.$$

Remark 6.1 It follows from more general result by May and Gugenheim (Corollary 5.17 in [19]) that $H^*(L_1)$ is generated by $H^1(L_1)$ with respect to matric Massey products.

The second part of the Buchstaber conjecture was not treated in [10], moreover it follows from the Proposition 4.3 that the $2q$-fold Massey product $\langle g_+^{q-1}, \underbrace{e^1, \ldots, e^1}_{2q-1} \rangle$ from the Theorem 6.1 is trivial.

In 2000 Buchstaber's PhD-student Artel'nykh considered the second part of the Buchstaber conjecture. In particular he claimed the following theorem.

Theorem 6.2 *[1] There are non-trivial Massey products*

$$g_-^q \in \mathbb{K}\langle \underbrace{e^2, \ldots, e^2}_{q-1}, g_+^{q-1}, e^1 \rangle, q \geq 2, \qquad g_+^{2l+1} \in \mathbb{K}\langle \underbrace{e^2, \ldots, e^2}_{3l+1}, g_+^{2l} \rangle, l \geq 1.$$

One can see that the cohomology classes g_+^{2l} were not represented by means of non-trivial Massey products. On the another hand Artel'nykh's article contain only a brief sketch of the proof.

We have mentioned that the Massey products in $H^*(\mathfrak{g})$ and $H^*(\mathrm{gr}_C\mathfrak{g})$ are closely related. Recall that $\mathrm{gr}_C L_1 \cong \mathfrak{m}_0$. We came to the problem of description of Massey products in the cohomology $H^*(\mathfrak{m}_0)$. The special question is the description of equivalency classes of trivial Massey products

$\langle\omega_1, \dots, \omega_n\rangle$ of 1-cohomology classes $\omega_1, \dots, \omega_n$. The purpose of this interest is to consider Massey products of the form $\langle\omega_1, \dots, \omega_n, \Omega\rangle$, where Ω is an element from $H^*(\mathfrak{g})$.

An infinite dimensional space $H^2(\mathfrak{m}_0)$ is spanned by the cohomology classes of following 2-cocycles [12]

$$\begin{aligned} &\omega(e^2 \wedge e^3) = e^2 \wedge e^3, \omega(e^3 \wedge e^4) = e^3 \wedge e^4 - e^2 \wedge e^5, \\ &\omega(e^4 \wedge e^5) = e^4 \wedge e^5 - e^3 \wedge e^6 + e^2 \wedge e^7, \dots, \\ &\omega(e^k \wedge e^{k+1}) = \sum_{l=0}^{k-2} (-1)^l e^{k-l} \wedge e^{k+1+l}, \dots \end{aligned} \tag{10}$$

All of the cocycles (10) can be represented as Massey products. Namely let consider the following matrix of a formal connection

$$A = \begin{pmatrix} 0 & e^2 & -e^3 & \dots & (-1)^k e^k & (-1)^{k+1} e^{k+1} & 0 \\ 0 & 0 & e^1 & 0 & \dots & 0 & e^{k+1} \\ 0 & 0 & 0 & e^1 & 0 & \dots & e^k \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & e^1 & e^3 \\ 0 & 0 & 0 & 0 & \dots & 0 & e^2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

For the related cocycle $c(A)$ we have

$$c(A) = \sum_{l=2}^{k+1} (-1)^l e^l \wedge e^{k+3-l} = 2\omega(e^k \wedge e^{k+1}).$$

Thus we have proved the following

Proposition 6.1

$$2\omega(e^k \wedge e^{k+1}) \in \langle e^2, \underbrace{e^1, \dots, e^1}_{2k-3}, e^2\rangle, \quad k \geq 2.$$

Example 6.1 We take $k = 2$ and the matrix A of a formal connection that corresponds to $\langle e^2, e^1, e^2\rangle$

$$A = \begin{pmatrix} 0 & e^2 & -e^3 & 0 \\ 0 & 0 & e^1 & e^3 \\ 0 & 0 & 0 & e^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The related cocycle $c(A)$ is equal to

$$c(A) = 2e^2 \wedge e^3 = 2\omega(e^2 \wedge e^3).$$

The space $H^1(\mathfrak{m}_0)$ is spanned by e^1 and e^2 and hence every n-fold Massey product of elements from $H^1(\mathfrak{m}_0)$ has a form

$$\underbrace{\langle \alpha_1 e^1 + \beta_1 e^2, \alpha_2 e^1 + \beta_2 e^2, \dots, \alpha_n e^1 + \beta_n e^2 \rangle}_{n}.$$

A product $e^1 \wedge e^2 = de^3$ is cohomologicaly trivial, hence a triple product

$$\langle \omega_1, \omega_2, \omega_3 \rangle = \langle \alpha_1 e^1 + \beta_1 e^2, \alpha_2 e^1 + \beta_2 e^2, \alpha_3 e^1 + \beta_3 e^2 \rangle$$

is defined for any choice of constants $\alpha_i, \beta_i \in \mathbb{K}, i = 1, 2, 3$.

$$A = \begin{pmatrix} 0 & \omega_1 & \gamma_1 e^3 & 0 \\ 0 & 0 & \omega_2 & \gamma_2 e^3 \\ 0 & 0 & 0 & \omega_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_1 = \alpha_1\beta_2 - \alpha_2\beta_1, \quad \gamma_2 = \alpha_2\beta_3 - \alpha_3\beta_2,$$

The related cocycle $c(A) = \gamma_2 \omega_1 \wedge e^3 - \gamma_1 \omega_3 \wedge e^3$ is trivial if and only if

$$\beta_1(\alpha_2\beta_3 - \alpha_3\beta_2) - \beta_3(\alpha_1\beta_2 - \alpha_2\beta_1) = 0. \tag{11}$$

We have mentioned above that if n-fold Massey product $\langle \omega_1, \omega_2, \dots, \omega_n \rangle$ is defined than all $(p+1)$-fold Massey products $\langle \omega_i, \omega_{i+1}, \dots, \omega_{i+p} \rangle$ for $1 \le i \le n-1, 1 \le p \le n-2, i+p \le n$ are defined and trivial.

Theorem 6.3 *Up to an equivalence the following trivial n-fold Massey products of non-zero cohomology classes from $H^1(\mathfrak{m}_0)$ are defined:*

name	*Massey product*	*parameters*
A_λ^{n+1}	$\underbrace{\langle \alpha e^1 + \beta e^2, \alpha e^1 + \beta e^2, \dots, \alpha e^1 + \beta e^2 \rangle}_{n}$	$n \ge 3,\ \lambda = (\alpha, \beta) \in \mathbb{K}P^1$
$B_{\alpha,\beta}^{n+1}$	$\underbrace{\langle \lambda_1 e^1 + e^2, \lambda_2 e^1 + e^2, \dots, \lambda_n e^1 + e^2 \rangle}_{n}$	$n \ge 3,\ \lambda_i = i\alpha + \beta,$ $\alpha, \beta \in \mathbb{K},\ \alpha \ne 0$
$C_{l,\alpha}^{n+1}$	$\langle \underbrace{e^1, \dots, e^1}_{l}, e^2 + \alpha e^1, \underbrace{e^1, \dots, e^1}_{n-l-1} \rangle$	$\alpha \in \mathbb{K},\ n \ge 3,$ $0 \le l \le n-1$
$D_{\alpha,\beta}^{2k+3}$	$\langle e^2 + \alpha e^1, \underbrace{e^1, \dots, e^1}_{2k}, e^2 + \beta e^1 \rangle$	$k \ge 1,\ \alpha, \beta \in \mathbb{K}$

Proof. The statement of the present theorem is equivalent to Benoist's classification [4] of indecomposable thread modules over $\mathfrak{m}_0$ ($\mathfrak{m}_0$ is graded as $\mathrm{gr}_C\mathfrak{m}_0 \cong \mathfrak{m}_0$). More precisely we consider a finite-dimensional $\mathfrak{m}_0$-module V with a basis $v_1, \ldots, v_{n+1}$ such that

$$e_1 v_i = \alpha_i v_{i-1},\ e_2 v_i = \beta_i v_{i-1},\ i = 1, \ldots, n+1.$$

In the last formula we assume that $v_0 = 0$. It is sufficient to define only an action of e_1 and e_2 on V because the algebra $\mathfrak{m}_0$ is generated by them.

Taking the corresponding matrix A of this representation with respect to the basis $v_1, \ldots, v_{n+1}$ we see that it has elements $\omega_i = \alpha_i e^1 + \beta_i e^2$ at its second diagonal. One can regard A as a defining system of the Massey product $\langle \underbrace{\omega_1, \ldots, \omega_n}_{n} \rangle, \omega_i = \alpha_i e^1 + \beta_i e^2$. Obviously V is decomposable as a direct sum of thread modules if and only if $e_1 v_i = e_2 v_i = 0$ for some $i, 1 \le i \le n+1$. The last condition means that $\omega_i = 0$ at the second diagonal of the matrix A.

We will prove this theorem by induction and start with triple products. First of all let consider the case when all $\beta_i \neq 0, i = 1, 2, 3$. Taking an equivalent defining system one can assume that $\beta_i = 1$, $i = 1, 2, 3$. In terms of representations it means that one can choose a base $v_1, \ldots, v_4$ of V such as

$$e_2 v_i = v_{i-1}, \quad i = 1, \ldots, 4.$$

The equation (11) in our case looks in a following way

$$2\alpha_2 - \alpha_1 - \alpha_3 = 0,$$

and it means that the numbers $\alpha_1, \alpha_2, \alpha_3$ form an arithmetic progression and we have either a type $B^4_{\alpha,\beta}$ or A^4_λ with $\lambda = (\alpha, 1)$. We keep Benoist's notations [4] of types of thread modules.

The equation (11) implies that if one constant from $\beta_1, \beta_2, \beta_3$ is equal to zero, then at least another one β_i is trivial also.

The case when $\beta_1 = \beta_2 = \beta_3 = 0$ is equivalent to $\langle e^1, e^1, e^1 \rangle$, i.e. to the type $A^4_{1,0}$ from the table above. The remaining three possibilities are $C^4_{l,\alpha}$, $l = 0, 1, 2$.

Let a n-fold product $\langle \omega_1, \ldots, \omega_n \rangle$ be defined and trivial then $(n-1)$-fold product $\langle \omega_1, \ldots, \omega_{n-1} \rangle$ is also trivial and by our inductive assumption is equivalent to some case from the table above.

The triple Massey product $\langle\omega_{n-2},\omega_{n-1},\omega_n\rangle$ is trivial in its turn and one can write out the equation (11) for all classes $\langle\omega_1,\dots,\omega_{n-1}\rangle$:

$\langle\omega_1,\dots,\omega_{n-1}\rangle$	equation (11) for $\langle\omega_{n-2},\omega_{n-1},\omega_n\rangle$	$\langle\omega_1,\dots,\omega_n\rangle$
$A^n_\lambda,\ \lambda=(\alpha,\beta)$	$\alpha\beta\beta_n=\beta^2\alpha_n$	$A^{n+1}_\lambda, C^{n+1}_{n-1,\rho}$
$B^n_{\alpha,\beta}$	$(\alpha n+\beta)\beta_n=\alpha_n$	$B^{n+1}_{\alpha,\beta}$
$C^n_{l,\alpha}$	$0=0,\ l<n{-}2;$ $2\beta_n=0,\ l=n{-}2;$ $-\beta_n=0,\ l=n{-}1.$	? $C^{n+1}_{l,\alpha}$ $C^{n+1}_{l,\alpha}$
$D^{2k+3}_{\alpha,\beta}$	$-\beta_n=0$	?

Lemma 6.1 *The following Massey products*

$$\langle e^2+\alpha e^1,\underbrace{e^1,\dots,e^1}_{2k},e^2+\beta e^1,e^1\rangle,\quad \langle e^1,e^2+\alpha e^1,\underbrace{e^1,\dots,e^1}_{2k},e^2+\beta e^1\rangle$$

are defined and non-trivial.

Proof. In order to simplify the notations we are going to consider only the first Massey product with $k=1$, the general case can be treated analogously. We will write only non-trivial entries of the defining system A (so-called the Massey triangle)

$$\begin{array}{ccccc}
e^2+\alpha\,e^1 & -e^3+\dots & e^4+\dots & (\alpha{-}\beta)\,e^5+\dots & \\
 & e^1 & 0+\dots & \dots & -3\,e^5+\dots \\
 & & e^1 & e^3+\dots & -2e^4+\dots \\
 & & & e^2+\beta e^1 & -e^3+\dots \\
 & & & & e^1
\end{array}$$

The related cocycle $c(A)$ is equal to

$$c(A)=3e^3{\wedge}e^4-3e^2{\wedge}e^5+\dots=3\omega(e^3{\wedge}e^4)+\dots,$$

where dots stand everywhere instead of summands of lower gradings.

Hence one can conclude that a trivial Massey product of type D is not extendable to a higher fold trivial Massey product. The same arguments show that $C^n_{l,\alpha}$ is extended only to $C^{n+1}_{l,\alpha}$ if $l>0$ and we have two possibilities in the case $C^{2k+1}_{0,\beta}$. They are $C^{2k+2}_{0,\beta}$ and $D^k_{\alpha,\beta}$ respectively.

Theorem 6.4 *The cohomology algebra $H^*(\mathfrak{m}_0)$ is generated with respect to the non-trivial Massey products by $H^1(\mathfrak{m}_0)$, namely*

$$\omega(e^2\wedge e^{i_2}\wedge\ldots\wedge e^{i_q}\wedge e^{i_q+1}) = e^2\wedge\omega(e^{i_2}\wedge\ldots\wedge e^{i_q}\wedge e^{i_q+1}),$$

$$2\omega(e^k\wedge e^{k+1}) \in \langle e^2, \underbrace{e^1,\ldots,e^1}_{2k-3}, e^2\rangle,$$

$$(-1)^{i_1}\omega(e^{i_1}\wedge e^{i_2}\wedge\ldots\wedge e^{i_q}\wedge e^{i_q+1}) \in \langle e^2, \underbrace{e^1,\ldots,e^1}_{i_1-2}, \omega(e^{i_2}\wedge\ldots\wedge e^{i_q}\wedge e^{i_q+1})\rangle,$$

Proof. First of all we present a graded defining system (a graded formal connection) A for a Massey product $\langle e^2, e^1, \ldots, e^1, \omega(e^{i_2}\wedge\ldots\wedge e^{i_q}\wedge e^{i_q+1})\rangle$. To simplify the formulas we will write ω instead of $\omega(e^{i_2}\wedge\ldots\wedge e^{i_q}\wedge e^{i_q+1})$.

One can verify that the following matrix A with non-zero entries at the second diagonal, first line and first row gives us an answer.

$$A = \begin{pmatrix} 0 & e^2 & -e^3 & e^4 & \ldots & (-1)^{i_1}e^{i_1} & 0 \\ 0 & 0 & e^1 & 0 & \ldots & 0 & D_{-1}^{i_1-2}\omega \\ 0 & 0 & 0 & e^1 & \ldots & 0 & D_{-1}^{i_1-3}\omega \\ & & & & \ldots & & \\ 0 & 0 & 0 & 0 & \ldots & e^1 & D_{-1}\omega \\ 0 & 0 & 0 & 0 & \ldots & 0 & \omega \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

The proof follows from

$$d(D_{-1}^k\omega) = e^1\wedge D_{-1}^{k-1}\omega, \quad d((-1)^k e^k) = (-1)^{k-1}e^{k-1}\wedge e^1.$$

The related cocycle $c(A)$ is equal to

$$c(A) = \sum_{l\geq 2}^{i_1}(-1)^l e^l \wedge D_{-1}^{i_1-l}\omega = (-1)^{i_1}\sum_{k\geq 0}^{i_1-2}(-1)^k D_1^k e^{i_1} \wedge D_{-1}^k\omega$$

Example 6.2 We take $\langle e^2, e^1, \omega(e^4\wedge e^5)\rangle$.

$$A = \begin{pmatrix} 0 & e^2 & -e^3 & 0 \\ 0 & 0 & e^1 & D_{-1}\omega(e^4\wedge e^5) \\ 0 & 0 & 0 & \omega(e^4\wedge e^5) \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

One can verify the computations of its related cocycle $c(A)$

$$\begin{aligned} c(A) &= e^2\wedge D_{-1}\omega(e^4\wedge e^5) - e^3\wedge\omega(e^4\wedge e^5) \\ &= e^2\wedge(e^4\wedge e^6 - 2e^3\wedge e^7 + 3e^2\wedge e^8) - e^3\wedge(e^4\wedge e^5 - e^3\wedge e^6 + e^2\wedge e^7) \\ &= -e^3\wedge e^4\wedge e^5 + e^2\wedge e^4\wedge e^6 - e^2\wedge e^3\wedge e^7 = -\omega(e^3\wedge e^4\wedge e^5). \end{aligned}$$

Lemma 6.2

$$\sum_{k\geq 0}(-1)^k D_1^k e^{i_1}\wedge D_{-1}^k\omega=\omega(e^{i_1}\wedge e^{i_2}\wedge\ldots\wedge e^{i_q}\wedge e^{i_q+1})$$

Proof. This lemma is almost evident. Both parts of the equality are closed forms. One has to compare the monomials of the form $e^{j_1}\wedge e^{j_2}\wedge\ldots\wedge e^{j_q}\wedge e^{j_q+1}$ in the decompositions of the left-hand and right-hand sides. The operator D_{-1} strictly increases the difference between two last superscripts of monomials

$$D_{-1}(e^{j_1}\wedge e^{j_2}\wedge\ldots\wedge e^{j_q}\wedge e^{j_q+1})=\sum_{k\geq 0}(-1)^k D_1^k(e^{j_1}\wedge e^{j_2}\wedge\ldots\wedge e^{j_q})\wedge D_{-1}^{k+1}e^{j_q+1}.$$

Hence there is the only one monomial of the form we are looking for on the left-hand side and the same one on the right-hand side and it is $e^{i_1}\wedge e^{i_2}\wedge\ldots\wedge e^{i_q}\wedge e^{i_q+1}$.

Lemma 6.3 *Let $\tilde{A}$ be an arbitrary defining system (the matrix of a formal connection) for $\langle e^2,e^1,\ldots,e^1,\omega(e^{i_2}\wedge\ldots\wedge e^{i_q}\wedge e^{i_q+1})\rangle$. Then its related cocycle $c(\tilde{A})$ is equal to*

$$(-1)^{i_1}\omega(e^{i_1}\wedge\ldots\wedge e^{i_q}\wedge e^{i_q+1})+\sum_{j_1<i_1}\lambda_{j_1,\ldots,j_q}\omega(e^{j_1}\wedge\ldots\wedge e^{j_q}\wedge e^{j_q+1})+e^1\wedge\Omega,$$

for some constants $\lambda_{j_1,\ldots,j_q}$ and q-form Ω.

Proof. We will rewrite our defining system $\tilde{A}$ in the form of a Massey triangle of the defining system $\tilde{A}$.

$$\begin{array}{cccccc}
e^2 & -e^3+\rho_1^1 & e^4+\rho_1^2 & \ldots & (-1)^{i_1}e^{i_1}+\rho_1^{i_1-2} & \\
 & e^1 & \rho_2^1 & \ldots & \ldots & D_{-1}^{i_1-2}\omega+\ldots+\Omega_{i_1-2} \\
 & & \ldots & \ldots & \ldots & \ldots \\
 & & & e^1 & \rho_{i_1-2}^1 & D_{-1}^2\omega+\ldots+\Omega_2 \\
 & & & & e^1 & D_{-1}\omega+\Omega_1 \\
 & & & & & \omega
\end{array}$$

Ω_i are some closed q-forms. 1-forms $\rho_1^1,\ldots,\rho_{i_1-2}^1$ standing at the second diagonal are also closed and therefore they are linear combinations of e^1 and e^2. Continue this procedure and using the Maurer-Cartan equation and an inductive assumption it is easy to see that ρ_k^l is a linear combination of $e^1,\ldots,e^{l+1}$. Hence the maximal value of superscript that we can meet at the first line of our triangle is i_1. Thus one can conclude that we have the only one monomial of the form $e^{i_1}\wedge\ldots\wedge e^j\wedge e^{j+1}$, $i_1<\ldots<j$ in the decomposition of the related cocycle $c(\tilde{A})$. It comes

from the summand $(-1)^{i_1}(e^{i_1}+\rho_1^{i_1-2})\wedge\omega$ in the formula of $c(\tilde{A})$ and it is $(-1)^{i_1}e^{i_1}\wedge\ldots\wedge e^{i_q}\wedge e^{i_q+1}$.

The Lemma 6.3 provides us with a proof of non-triviality of the Massey products $\langle e^2, e^1, \ldots, e^1, \omega(e^{i_2}\wedge\ldots\wedge e^{i_q}\wedge e^{i_q+1})\rangle$. The proof in the case $\langle e^2, e^1, \ldots, e^1, e^2\rangle$ can be obtained by the same arguments.

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