ERROR ESTIMATION FOR THE DIRECT ALGORITHM OF PROJECTIVE MAPPING CALCULATION IN MULTIPLE VIEW GEOMETRY *

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Error estimate for the direct algorithm of determination of the projective mapping is presented. Such estimates are important in the problem of gluing of 2D projective images obtained from different points in the space [1], [2], [5]. The obtained estimate suggests that the direct algorithm of projective mapping calculation is rather accurate and robust.

1. Introduction

The problem of gluing 2D projective images obtained by central projection from different points in space becomes important in modern computer geometry due to the rapid development of digital image technologies. See, for example, [1], [5], [6], [7], [8], [9], [10], [11].

From geometrical point of view this problem can be formulated as a problem of computation of projective mapping F_P which bounds two domains D_1 and D_2 placed in the same affine coordinate map of a projective plane RP^2. In order to solve this problem in is necessary: 1) to recognize large enough quantity of points (*conjugate points*) which correspond to each other by

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means of unknown projective mapping F_P; 2) to create a robust algorithm for F_P calculation and to estimate possible distortion of this calculation.

In modern multiple-view geometry there is a high demand for development of new efficient computer algorithms for conjugate points recognition (such algorithms are called *tracking algorithms* or simply *trackers*). A new approach to the tracking problem based on ideas from [3], [4] was suggested in our previous paper [5]. The next step is to choose a robust algorithm to calculate the projective mapping. It should be stressed that in most cases conjugate points are determined by trackers only approximately. That is why the problem of the accuracy and stability of projective mapping calculation based on perturbed conjugate points is very important.

In the Theorem, proved in the next section, the error estimate for projective mapping is obtained in terms of special matrix representation for the mapping itself and it's error [5]. The maximal absolute value of matrix elements for such error representation is estimated. It enables easy estimation of local distortion in any given part of the screen.

Below we will use the following vector norm in $\mathbf{R}^n$ and corresponding operator norm in the space of $n \times n$ matrixes:

$$\|\mathbf{x}\| = \max_{1 \le i \le n} |x_i| \quad \forall\, \mathbf{x} = (x_1, \dots, x_n) \in \mathbf{R}^n, \tag{1}$$

$$\|C\| = \max_{1 \le i \le n} \sum_{j=1}^{n} |c_{ij}| \quad \forall\, C = (c_{ij})_{n \times n}. \tag{2}$$

In order to formulate the main result of the paper we need to describe the direct algorithm of projective mapping calculation and define the fixed matrix representation for the projective mapping.

Assume that unknown projective mapping F_P maps 4 points $P, Q, R, T \in RP^2$ situated in general position (see [2]) into 4 points $P', Q', R', T' \in RP^2$ also situated in general position respectively:

$$F(P) = P', \quad F(Q) = Q', \quad F(R) = R', \quad F(T) = T',$$

and F_P is represented by a set of unknown variables (f_{ij}) $(1 \le i, j \le 3)$ which are organized in a square 3×3 matrix F of a linear operator $R^3 \to R^3$ which corresponds to the mapping F_P. In order to define (f_{ij}) uniquely we will assume that all 8 points $\{P, Q, R, T\}$ and $\{P', Q', R', T'\}$ belong to the affine map $S_3 = \{(x_1, x_2, x_3) : x_3 \ne 0\} \subset RP^2$ and their coordinates are represented by the following 3-dimensional vectors:

$$\begin{aligned} P &= (p_1, p_2, 1), & Q &= (q_1, q_2, 1), \\ R &= (r_1, r_2, 1), & T &= (t_1, t_2, 1), \\ P' &= (p_1', p_2', 1), & Q' &= (q_1', q_2', q_3'), \\ R' &= (r_1', r_2', r_3'), & T' &= (t_1', t_2', t_3'). \end{aligned} \tag{3}$$

This assumption is not restrictive in our case because all points of a photographic image always belong to the same affine map in RP^2. Under this assumption the matrix representation

$$(f_{ij}) \quad (1 \le i, j \le 3)$$

of the projective mapping F_P is unique [5].

Below we will use the notations:

$$\begin{aligned} a_p &= p_1', \quad b_p = p_2', & a_q &= \frac{q_1'}{q_3'}, \quad b_q = \frac{q_2'}{q_3'}, \\ a_r &= \frac{r_1'}{r_3'}, \quad b_r = \frac{r_2'}{r_3'}, & a_t &= \frac{t_1'}{t_3'}, \quad b_t = \frac{t_2'}{t_3'}. \end{aligned} \tag{4}$$

Consider the system of linear equations [5]

$$A\mathbf{x} = \mathbf{y}, \tag{5}$$

where $\mathbf{x}$ is vector of unknown variables

$$\mathbf{x} = (f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}, q_3', r_3', t_3')^T, \tag{6}$$

and $A, \mathbf{y}$ are given by:

$$\mathbf{y} = (a_p, 0, 0, 0, b_p, 0, 0, 0, 1, 0, 0, 0)^T, \tag{7}$$

$$A = \begin{pmatrix} p_1 & p_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_1 & q_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -a_q & 0 & 0 \\ r_1 & r_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_r & 0 \\ t_1 & t_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_t \\ 0 & 0 & 0 & p_1 & p_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_1 & q_2 & 1 & 0 & 0 & 0 & -b_q & 0 & 0 \\ 0 & 0 & 0 & r_1 & r_2 & 1 & 0 & 0 & 0 & 0 & -b_r & 0 \\ 0 & 0 & 0 & t_1 & t_2 & 1 & 0 & 0 & 0 & 0 & 0 & -b_t \\ 0 & 0 & 0 & 0 & 0 & 1 & p_1 & p_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_1 & q_2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_1 & r_2 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_1 & t_2 & 1 & 0 & 0 & -1 \end{pmatrix} \tag{8}$$

The system (5) consists of 12 linear scalar equations with 12 unknown variables: 9 elements f_{ij} of the projective mapping and 3 unknown point coordinates q'_3, r'_3, t'_3 (note that the matrix A contains known values only). The system (5) is determined by two 4-sets of conjugate points $\{P, Q, R, T\}$, $\{P', Q', R', T'\}$ in RP^2. If the points in both sets are situated in general position then, according to Lemma 3.1 from [5], matrix A is non-degenerate and it's determinate is proportional to the product of the areas of triangles $\triangle PQR, \triangle PRT, \triangle PQT$, and $\triangle Q'R'T'$:

$$|detA| = 16 S_{\triangle PQR} S_{\triangle PRT} S_{\triangle PQT} S_{\triangle Q'R'T'}. \tag{9}$$

Direct algorithm consists in determination of projective mapping F_P by means of numerical solution of the system (5). In the next Section we analyze the stability of this algorithm by estimating the error in matrix F calculation resulting from the possible errors in the coordinates of given conjugate points P, Q, R, T and P', Q', R', T'.

2. Error estimation

We assume that the given affine coordinates of the points P, Q, R, T and P', Q', R', T' are determined with possible errors bounded by the given constant δ. In applications the value of δ is determined, in particular, by the resolution of the digital images which are analyzed and the accuracy of the tracking algorithms. More precisely, we assume that instead of the true points P, Q, R, T and P', Q', R', T', the perturbed sets of points $\widetilde{P}, \widetilde{Q}, \widetilde{R}, \widetilde{T}$ and $\widetilde{P'}, \widetilde{Q'}, \widetilde{R'}, \widetilde{T'}$ are given:

$$\begin{aligned}
\widetilde{P} &= (p_1 + \Delta p_1, p_2 + \Delta p_2), & \widetilde{Q} &= (q_1 + \Delta q_1, q_2 + \Delta q_2), \\
\widetilde{R} &= (r_1 + \Delta r_1, r_2 + \Delta r_2), & \widetilde{T} &= (t_1 + \Delta t_1, t_2 + \Delta t_2), \\
\widetilde{P'} &= (a_p + \Delta a_p, b_p + \Delta b_p), & \widetilde{Q'} &= (a_q + \Delta a_q, b_q + \Delta b_q), \\
\widetilde{R'} &= (a_r + \Delta a_r, b_r + \Delta b_r), & \widetilde{T'} &= (a_t + \Delta a_t, b_t + \Delta b_t),
\end{aligned} \tag{10}$$

where

$$|\Delta p_i|, |\Delta q_i|, |\Delta r_i|, |\Delta t_i| < \delta, \ \forall i = 1,2;$$
$$|\Delta a_j|, |\Delta b_j| < \delta, \ \forall j \in \{p,q,r,t\}. \tag{11}$$

In applications the size of the screen is usually bounded by a given constant, so it is not restrictive to assume that all coordinates of the given conjugate points are bounded. If the coordinate origin is placed in the center of the square-shaped screen with the side length N, then all coordinates of the given perturbed points $\widetilde{P}, \widetilde{Q}, \widetilde{R}, \widetilde{T}$ as well as the coordinates of the unknown true points $\widetilde{P'}, \widetilde{Q'}, \widetilde{R'}, \widetilde{T'}$ are bounded by the value

$$\frac{N}{2}: \quad |a_j|, |b_j| \le \frac{N}{2}, \ \forall j \in \{p,q,r,t\}; \quad |p_i|, |q_i|, |r_i|, |t_i| \le \frac{N}{2}, \ \forall i = 1,2.$$

If only perturbed coordinates of conjugate points are known then coefficients of the system (5) are unknown. In this situation instead of (5) the following perturbed system of linear equations is used for the determination of the perturbed vector $(\mathbf{x} + \Delta\mathbf{x})$:

$$(A + \Delta A)(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{y} + \Delta\mathbf{y} \tag{12}$$

where

$$\Delta\mathbf{x} = (\Delta f_{11}, \Delta f_{12}, \ldots, \Delta f_{33}, \Delta q_3', \Delta r_3', \Delta t_3')^T, \tag{13}$$

$$\Delta\mathbf{y} = (\Delta a_p, \ 0, \ 0, \ 0, \ \Delta b_p, \ 0, \ 0, \ 0, \ 0, \ 0, \ 0, \ 0)^T,$$

$$\Delta A = \begin{pmatrix}
\Delta p_1 & \Delta p_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Delta q_1 & \Delta q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\Delta a_q & 0 & 0 \\
\Delta r_1 & \Delta r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\Delta a_r & 0 \\
\Delta t_1 & \Delta t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\Delta a_t \\
0 & 0 & 0 & \Delta p_1 & \Delta p_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Delta q_1 & \Delta q_2 & 0 & 0 & 0 & 0 & -\Delta b_q & 0 & 0 \\
0 & 0 & 0 & \Delta r_1 & \Delta r_2 & 0 & 0 & 0 & 0 & 0 & -\Delta b_r & 0 \\
0 & 0 & 0 & \Delta r_1 & \Delta r_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\Delta b_t \\
0 & 0 & 0 & 0 & 0 & 0 & \Delta p_1 & \Delta p_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Delta q_1 & \Delta q_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Delta r_1 & \Delta r_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Delta t_1 & \Delta t_2 & 0 & 0 & 0 & 0
\end{pmatrix}$$

We will assume that the configuration of conjugate points was chosen in such a way that the determinant of perturbed matrix $A + \Delta A$ is still separated from zero (according to (9) this assumption is quite realistic) :

$$det(A + \Delta A) > 0. \tag{14}$$

From (5), (12) and (14) it can be easily derived that

$$\Delta \mathbf{x} = (A + \Delta A)^{-1}(\Delta \mathbf{y} - \Delta A A^{-1} \mathbf{y}). \tag{15}$$

Summarizing the above-made assumptions, we consider the following situation:

1) two sets of 4 points each $\{P, Q, R, T\}$, $\{P', Q', R', T'\}$ in RP^2 belong to the affine map

$$S_3 = \{(x_1, x_2, x_3) : x_3 \neq 0\}$$

and their affine coordinates are given by (3);

2) $\{P, Q, R, T\}$ is mapped into $\{P', Q', R', T'\}$ by a projective mapping F_P which is represented by the uniquely determined 3×3 matrix $F = (f_{ij})$ which elements coincide with the first 9 components of the vector $\mathbf{x}$ – solution of the system of linear equations (5);

3) the perturbed set of points $\{\widetilde{P}, \widetilde{Q}, \widetilde{R}, \widetilde{T}\}$ is mapped into the perturbed set of points $\{\widetilde{P}', \widetilde{Q}', \widetilde{R}', \widetilde{T}'\}$ by the projective mapping $\widetilde{F}_P$ represented by the uniquely determined 3×3 matrix

$$\widetilde{F} = (f_{ij} + \Delta f_{ij})$$

which elements coincide with the first 9 components of the vector $\mathbf{x} + \Delta \mathbf{x}$ – solution of the perturbed system of linear equations (12);

4) affine coordinates of the perturbed points $\{\widetilde{P}, \widetilde{Q}, \widetilde{R}, \widetilde{T}\}$, $\{\widetilde{P}', \widetilde{Q}', \widetilde{R}', \widetilde{T}'\}$ are given by (10) with the errors in coordinates bounded by the constant $\delta > 0$ according to (11);

Let us denote the error of the projective mapping F_P representation, obtained from the solution of the perturbed system (12) by

$$\Delta F = \widetilde{F} - F = (\Delta f_{ij}).$$

Then it is easy to see that specified by the error ΔF distortion of the affine coordinates (p'_1, p'_2) of the point P' is bounded by

$$|\Delta p'_k| \leq (|p_1| + |p_2|) \max_{1\leq i,j\leq 2} |\Delta f_{ij}|, \qquad (1 \leq k \leq 2). \tag{16}$$

The upper bound for the right-hand expression in (16) is given by the following Theorem.

Theorem 2.1 *Assume that all the affine coordinates of all 16 points given by (3) and (10) are bounded in their absolute values by constants* $\frac{N}{2}$ *for first coordinate and* $\frac{M}{2}$ *for second coordinate. Then*

$$|\Delta f_{ij}| \leq 22.25\,\delta \left(\frac{N^{15} + M^{15}}{2S}\right) \qquad (1 \leq i, j \leq 2) \tag{17}$$

where

$$S = 256 S_{\Delta PQR} S_{\Delta PRT} S_{\Delta PQT} S_{\Delta Q'R'T'} S_{\Delta \widetilde{P}\widetilde{Q}\widetilde{R}} S_{\Delta \widetilde{P}\widetilde{R}\widetilde{T}} S_{\Delta \widetilde{P}\widetilde{Q}\widetilde{T}} S_{\Delta \widetilde{Q}'\widetilde{R}'\widetilde{T}'}.$$

Proof. From (15) it follows that

$$\|\Delta \mathbf{x}\| \leq \left\|(A + \Delta A)^{-1}\Delta \mathbf{y}\right\| + \left\|(A + \Delta A)^{-1}\Delta A A^{-1}\mathbf{y})\right\|. \tag{18}$$

Massive but direct calculation of the both terms in the right side of (18) in terms of the norm (2), and direct estimation of the first 9 components of the vector $\Delta \mathbf{x}$ (which according to (13) coincide with the matrix elements of the error matrix ΔF) using the relation (9) give, in particular, the inequality (17). $\square$

Corollary 2.1 *Let all assumptions of Theorem 2.1 hold and assume additionally that* $N = M$ *and the set of conjugate points was chosen in such a way that for some* $0 < \varepsilon < 1$

$$S_{\Delta PQR},\ S_{\Delta PRT},\ S_{\Delta PQT},\ S_{\Delta Q'R'T'} \geq (1-\varepsilon)\frac{N^2}{2}\,,$$

$$S_{\Delta \widetilde{P}\widetilde{Q}\widetilde{R}},\ S_{\Delta \widetilde{P}\widetilde{R}\widetilde{T}},\ S_{\Delta \widetilde{P}\widetilde{Q}\widetilde{T}},\ S_{\Delta \widetilde{Q}'\widetilde{R}'\widetilde{T}'} \geq (1-\varepsilon)\frac{N^2}{2}\,.$$

Then

$$|\Delta p'_k| \leq \frac{22.25\ \delta}{(1-\varepsilon)^8}, \qquad (1 \leq k \leq 2)\ .$$

In particular if $\varepsilon \approx 0$, which corresponds to the optimal position of four conjugate points in the four different corners of the screen, then the above inequality can be approximated by the following one

$$|\Delta p'_k| < 22.25\ \delta, \qquad (1 \leq k \leq 2).$$

The presented estimates show that the direct algorithm of projective mapping calculation in the problem of gluing digital images is sufficiently accurate and robust. These estimates also suggest the following simple rule of conjugate points configuration choice in order make the calculations more accurate: the product of squares of triangles built from the triples of such points must be as large as possible.

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