TOPOLOGY OF KEMPF–NESS SETS FOR ALGEBRAIC TORUS ACTIONS *

TARAS PANOV †

Department of Geometry and Topology, Faculty of Mathematics and Mechanics, Moscow State University, Leninskiye Gory, Moscow 119992, Russia

AND

Institute for Theoretical and Experimental Physics, Moscow 117259, Russia

E-mail: tpanov@mech.math.msu.su

In the theory of algebraic group actions on affine varieties, the concept of a Kempf–Ness set is used to replace the categorical quotient by the quotient with respect to a maximal compact subgroup. By making use of the recent achievements of "toric topology" we show that an appropriate notion of a Kempf–Ness set exists for a class of algebraic torus actions on quasiaffine varieties (coordinate subspace arrangement complements) arising in the "geometric invariant theory" approach to toric varieties. We proceed by studying the cohomology of these Kempf–Ness sets.

1. Introduction

The concept of a Kempf–Ness set plays an important role in the Geometric Invariant Theory, as explained, for example, in [16]. Given an affine variety S over $\mathbb{C}$ with an action of a reductive group G, one can find a compact subset $KN \subset S$ such that the categorical quotient $S/\!\!/\mathrm{G}$ is homeomorphic to the quotient KN/K of KN by a maximal compact subgroup $\mathrm{K} \subset \mathrm{G}$. Another important property of the Kempf–Ness set KN is that it is a K-equivariant deformation retract of S.

Our aim here is to extend the notion of a Kempf–Ness set to a class of algebraic torus actions on complex quasiaffine varieties (coordinate subspace arrangement complements) arising in the theory of toric varieties. Although

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our Kempf–Ness sets cannot be defined exactly in the same way as in the affine case, they possess the two above characteristic properties. In the case of projective toric variety, our Kempf–Ness set can be identified with the level surface for the moment map corresponding to a compact torus action on the complex space (see §4 of [9] for the definition of the corresponding moment maps). The toric Kempf–Ness sets also constitute a particular subclass of *moment-angle complexes*, toric spaces introduced in [10] and studied in detail in [6], [15], etc. We therefore consider our current approach to Kempf–Ness sets for algebraic torus actions as an attempt to establish a bridge between the topological and combinatorial study of torus actions, now known as "Toric Topology", and Geometric Invariant Theory.

In Section 2 we review the notion of Kempf–Ness sets for reductive groups acting on affine varieties. In Section 3 we outline the "geometric invariant theory" approach to toric varieties (due to Batyrev, Cox, etc.) as quotients of algebraic torus actions on coordinate subspace arrangement complements, and give a construction of a Kempf–Ness set using our "toric topology" construction of moment-angle complexes. In Section 4 we restrict our attention to torus actions arising from normal fans of convex polytopes. In this case the corresponding Kempf–Ness set admits a transparent geometric interpretation as a complete intersection of real quadratic hypersurfaces. The quotient toric variety is projective, and the Kempf–Ness set is the level surface for an appropriate moment map, thereby extending the analogy with the affine case even further in Section 5. In the last Section 6 we give a description of the cohomology ring of the Kempf–Ness set. As it is clear from an example provided, our Kempf–Ness sets may be quite complicated topologically; many interesting phenomena occur even for the torus actions corresponding to simple 3-dimensional fans.

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2. Kempf–Ness sets for affine varieties

We start by briefly reviewing quotients of reductive groups and Kempf–Ness sets, following [16].

Let G be a reductive algebraic group, and S an affine G-variety. Denote $S/\!\!/\mathrm{G}$ the complex affine variety corresponding to the subalgebra $\mathbb{C}[S]^{\mathrm{G}}$ of G-invariant polynomial functions on S, and let $\pi_{S,\mathrm{G}}\colon S \to S/\!\!/\mathrm{G}$ be the morphism dual to the inclusion $\mathbb{C}[S]^{\mathrm{G}} \to \mathbb{C}[S]$. Then $\pi_{S,\mathrm{G}}$ is surjective and establishes a bijection between closed G-orbits of S and points of $S/\!\!/\mathrm{G}$. Moreover, $\pi_{S,\mathrm{G}}$ is universal in the class of morphisms from S constant on G-orbits in the category of algebraic varieties (which explains the term "categorical quotient").

Let $\rho\colon \mathrm{G} \to \mathrm{GL}(V)$ be a representation of G, let K be a maximal compact subgroup of G, and let $\langle\ ,\ \rangle$ be a K-invariant hermitian form on V with associated norm $\|\ \|$. Given $v \in V$, consider the function $F_v\colon \mathrm{G} \to \mathbb{R}$ sending g to $\frac{1}{2}\|gv\|^2$. It has a critical point if and only if Gv is closed, and all critical points of F_v are minima, see (4.2) in [16]. Define the subset $KN \subset V$ by one of the following equivalent conditions:

$$\begin{aligned} KN &= \{v \in V\colon (dF_v)_e = 0\} \qquad (e \in \mathrm{G} \text{ is the unit}) \\ &= \{v \in V\colon T_v\mathrm{G}v \perp v\} \\ &= \{v \in V\colon \langle \gamma v, v\rangle = 0 \text{ for all } \gamma \in \mathfrak{g}\} \\ &= \{v \in V\colon \langle \kappa v, v\rangle = 0 \text{ for all } \kappa \in \mathfrak{k}\}, \end{aligned} \tag{2.1}$$

where $\mathfrak{g}$ (resp. $\mathfrak{k}$) is the Lie algebra of G (resp. K) and we consider $\mathfrak{k} \subseteq \mathfrak{g} \subseteq \mathrm{End}(V)$. Therefore, any point $v \in KN$ is a closest point to the origin in its orbit Gv. Then KN is called the *Kempf–Ness set* of V.

We may assume that the affine G-variety S is equivariantly embedded as a closed subvariety in a representation V of G. Then the *Kempf–Ness set* KN_S of S is defined as $KN \cap S$.

The importance of Kempf–Ness sets for the study of orbit quotients is due to the following result, which is proved as (4.7) and (5.1) in [16].

Theorem 2.1 (a) *The composition* $KN_S \hookrightarrow S \to S/\!\!/\mathrm{G}$ *is proper and induces a homeomorphism* $KN_S/\mathrm{K} \to S/\!\!/\mathrm{G}$.

(b) [Neeman] *There is a* K*-equivariant deformation retraction of* S *to* KN_S.

3. Algebraic torus actions

Let $N \cong \mathbb{Z}^n$ be an integral lattice of rank n, and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ the ambient real vector space. A convex subset $\sigma \in N_{\mathbb{R}}$ is called a *cone* if there exist vectors $a_1, \ldots, a_k \in N$ such that

$$\sigma = \{\mu_1 a_1 + \ldots + \mu_k a_k \colon \mu_i \in \mathbb{R},\ \mu_i \geqslant 0\}.$$

If the set $\{a_1, \ldots, a_k\}$ is minimal, then it is called the *generator set* of σ. A cone is called *strongly convex* if it contains no line through the origin; all the cones below are assumed to be strongly convex. A cone σ is called *regular* (resp. *simplicial*) if $a_1, \ldots, a_k$ can be chosen to form a subset of a $\mathbb{Z}$-basis of N (resp. an $\mathbb{R}$-basis of $N_{\mathbb{R}}$). A *face* of a cone σ is the intersection $\sigma \cap H$ with a hyperplane H for which the whole σ is contained in one of the two closed half-spaces determined by H; a face of a cone is again a cone. Every generator of σ spans a one-dimensional face, and every face of σ is spanned by a subset of the generator set.

A finite collection $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of cones in $N_{\mathbb{R}}$ is called a *fan* if a face of every cone in Σ belongs to Σ and the intersection of any two cones in Σ is a face of each. A fan Σ is called *regular* (resp. *simplicial*) if every cone in Σ is regular (resp. simplicial). A fan $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ is called *complete* if $N_{\mathbb{R}} = \sigma_1 \cup \ldots \cup \sigma_s$.

Let $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$ be the multiplicative group of complex numbers, and S^1 be the subgroup of complex numbers of absolute value one. The *algebraic torus* $\mathrm{T}_{\mathbb{C}} = N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$ is a commutative complex algebraic group with a maximal compact subgroup $\mathrm{T} = N \otimes_{\mathbb{Z}} \mathrm{S}^1 \cong (\mathrm{S}^1)^n$, the (compact) *torus*. A *toric variety* is a normal algebraic variety X containing the algebraic torus $\mathrm{T}_{\mathbb{C}}$ as a Zariski open subset in such a way that the natural action of $\mathrm{T}_{\mathbb{C}}$ on itself extends to an action on X.

There is a classical construction (cf. [3]) establishing a one-to-one correspondence between fans in $N_{\mathbb{R}}$ and complex n-dimensional toric varieties. Regular fans correspond to non-singular varieties, while complete fans give rise to compact ones. Below we review another construction of toric varieties as certain algebraic quotients; it is due to several authors (cf. [4], [8]).

In the rest of this section we assume that the one-dimensional cones of Σ span $N_{\mathbb{R}}$ as a vector space. Assume that Σ has m one-dimensional cones. We order them arbitrarily and consider the map $\mathbb{Z}^m \to N$ sending the ith generator of $\mathbb{Z}^m$ to the integer primitive vector a_i generating the ith one-dimensional cone. The corresponding map of the algebraic tori fits into an exact sequence

$$1 \longrightarrow \mathrm{G} \longrightarrow (\mathbb{C}^*)^m \longrightarrow \mathrm{T}_{\mathbb{C}} \longrightarrow 1, \tag{3.1}$$

where G is isomorphic to a product of $(\mathbb{C}^*)^{m-n}$ and a finite group. If Σ is a regular fan and has at least one n-dimensional cone, then $G \cong (\mathbb{C}^*)^{m-n}$. We also have an exact sequence of the corresponding maximal compact subgroups:

$$1 \longrightarrow \mathrm{K} \longrightarrow T^m \longrightarrow \mathrm{T} \longrightarrow 1 \tag{3.2}$$

(here and below we denote $T^m = (\mathrm{S}^1)^m$).

We say that a subset $\{i_1, \ldots, i_k\} \in [m] = \{1, \ldots, m\}$ is a *g-subset* if $\{a_{i_1}, \ldots, a_{i_k}\}$ is a subset of the generator set of a cone in Σ. The collection of g-subsets is closed with respect to the inclusion, and therefore forms an (abstract) simplicial complex on the set $[m]$, which we denote $\mathcal{K}_\Sigma$. Note that if Σ is a complete simplicial fan, then $\mathcal{K}_\Sigma$ is a triangulation of an $(n-1)$-dimensional sphere. Given a cone $\sigma \in \Sigma$, we denote by $g(\sigma) \subseteq [m]$ the set of its generators. Now set

$$A(\Sigma) = \bigcup_{\{i_1,\ldots,i_k\} \text{ is not a } g\text{-subset}} \{z \in \mathbb{C}^m \colon z_{i_1} = \ldots = z_{i_k} = 0\}$$

and

$$U(\Sigma) = \mathbb{C}^m \setminus A(\Sigma).$$

Both sets depend only on the combinatorial structure of the simplicial complex $\mathcal{K}_\Sigma$; the set $U(\Sigma)$ coincides with the *coordinate subspace arrangement complement* $U(\mathcal{K}_\Sigma)$ considered in §8.2 of [6].

The set $A(\Sigma)$ is an affine variety, while its complement $U(\Sigma)$ admits a simple affine cover, as described in the following statement.

Proposition 3.1 *Given a cone $\sigma \in \Sigma$, set $z^{\hat{\sigma}} = \prod_{j \notin g(\sigma)} z_j$ and define*

$$V(\Sigma) = \{z \in \mathbb{C}^m \colon z^{\hat{\sigma}} = 0 \textit{ for all } \sigma \in \Sigma\}$$

and

$$U(\sigma) = \{z \in \mathbb{C}^m \colon z_j \neq 0 \textit{ if } j \notin g(\sigma)\}.$$

Then $A(\Sigma) = V(\Sigma)$ and

$$U(\Sigma) = \mathbb{C}^m \setminus V(\Sigma) = \bigcup_{\sigma \in \Sigma} U(\sigma).$$

Proof. We have

$$\mathbb{C}^m \setminus V(\Sigma) = \bigcup_{\sigma \in \Sigma} \{z \in \mathbb{C}^m \colon z^{\hat{\sigma}} \neq 0\} = \bigcup_{\sigma \in \Sigma} U(\sigma).$$

On the other hand, given a point $z \in \mathbb{C}^m$, denote by $\omega(z) \subseteq [m]$ the set of its zero coordinates. Then $z \in \mathbb{C}^m \setminus A(\Sigma)$ if and only if $\omega(z)$ is a g-subset.

This is equivalent to saying that $z \in U(\sigma)$ for some $\sigma \in \Sigma$. Therefore, $\mathbb{C}^m \setminus A(\Sigma) = \cup_{\sigma\in\Sigma} U(\sigma)$, thus proving the statement.

The complement $U(\Sigma)$ is invariant with respect to the $(\mathbb{C}^*)^m$-action, and it is easy to see that the subgroup G from (3.1) acts on $U(\Sigma)$ with finite isotropy subgroups if Σ is simplicial (or even freely if Σ is a regular fan). The corresponding quotient is identified with the toric variety X_Σ determined by Σ. The more precise statement is as follows (it is proved as Theorem 2.1 in [8]).

Theorem 3.2 *Assume that the one-dimensional cones of Σ span $N_{\mathbb{R}}$ as a vector space.*

(a) *The toric variety X_Σ is naturally isomorphic to the categorical quotient of $U(\Sigma)$ by* G.

(b) *X_Σ is the geometric quotient of $U(\Sigma)$ by* G *if and only if Σ is simplicial.*

Therefore, if Σ is a simplicial (in particular, regular) fan satisfying the assumption of Theorem 3.2, then all the orbits of the G-action on $U(\Sigma)$ are closed and the categorical quotient $U(\Sigma)/\!\!/\mathrm{G}$ can be identified with $U(\Sigma)/\mathrm{G}$. However, the analysis of the previous section does not apply here, as $U(\Sigma)$ is *not* an affine variety in $\mathbb{C}^m$ (it is only quasiaffine in general). For example, if Σ is a complete fan, then the G-action on the whole $\mathbb{C}^m$ has only one closed orbit, the origin, and the quotient $\mathbb{C}^m/\!\!/\mathrm{G}$ consists of a single point. In the rest of the paper we show that an appropriate notion of the Kempf–Ness set exists for this class of torus actions, and study some of its most important topological properties.

Consider the unit polydisc

$$(\mathrm{D}^2)^m = \{z \in \mathbb{C}^m \colon |z_j| \leqslant 1 \text{ for all } j\}.$$

Given $\sigma \in \Sigma$, define

$$\mathcal{Z}(\sigma) = \{z \in (\mathrm{D}^2)^m \colon |z_j| = 1 \text{ if } j \notin g(\sigma)\},$$

and

$$\mathcal{Z}(\Sigma) = \bigcup_{\sigma\in\Sigma} \mathcal{Z}(\sigma).$$

The subset $\mathcal{Z}(\Sigma) \subseteq (\mathrm{D}^2)^m$ is invariant with respect to the T^m-action. (We have $\mathcal{Z}(\Sigma) = \mathcal{Z}_{\mathcal{K}_\Sigma}$, where $\mathcal{Z}_{\mathcal{K}}$ is the *moment-angle complex* associated with a simplicial complex $\mathcal{K}$ in §6.2 of [6].) Note that $\mathcal{Z}(\sigma) \subset U(\sigma)$, and therefore, $\mathcal{Z}(\Sigma) \subset U(\Sigma)$ by Proposition 3.1.

Proposition 3.3 *Assume that Σ is a complete simplicial fan. Then $\mathcal{Z}(\Sigma)$ is a compact T^m-manifold of dimension $m+n$.*

Proof. As $\mathcal{K}_\Sigma$ is a triangulation of an $(n-1)$-dimensional sphere, the result follows from Lemma 6.13 of [6] (or Lemma 3.3 of [15]).

Theorem 3.4 *Assume that Σ is a simplicial fan.*

(a) *If Σ is complete, then the composition $\mathcal{Z}(\Sigma) \hookrightarrow U(\Sigma) \to U(\Sigma)/\mathrm{G}$ induces a homeomorphism $\mathcal{Z}(\Sigma)/\mathrm{K} \to U(\Sigma)/\mathrm{G}$.*

(b) *There is a T^m-equivariant deformation retraction of $U(\Sigma)$ to $\mathcal{Z}(\Sigma)$.*

Proof. Denote by $\operatorname{cone}\mathcal{K}'_\Sigma$ the cone over the barycentric subdivision of $\mathcal{K}_\Sigma$ and by $C(\Sigma)$ the topological space $|\operatorname{cone}\mathcal{K}'_\Sigma|$ with the dual *face decomposition*, see §3.1 of [15] for details. (If Σ is a complete fan, then $\mathcal{K}_\Sigma$ is a sphere triangulation, $C(\Sigma)$ can be identified with the unit ball in $N_\mathbb{R}$, and the face decomposition of its boundary is Poincaré dual to $\mathcal{K}_\Sigma$). The space $C(\Sigma)$ has a face $C(\sigma)$ of dimension $n-g(\sigma)$ for each cone $\sigma \in \Sigma$. Set

$$T(\sigma) = \{(t_1, \ldots, t_m) \in T^m \colon t_i = 1 \text{ if } j \notin g(\sigma)\}.$$

This is a $g(\sigma)$-dimensional coordinate subgroup in T^m. As detailed in [10] and §3.1 of [15], the set $\mathcal{Z}(\Sigma)$ can be described as the identification space

$$\mathcal{Z}(\Sigma) = \big(T^m \times C(\Sigma)\big)/\!\sim,$$

where $(t,x) \in T^m \times C(\Sigma)$ is identified with $(s,x) \in T^m \times C(\Sigma)$ if $x \in C(\sigma)$ and $t^{-1}s \in T(\sigma)$ for some $\sigma \in \Sigma$. The map of tori $T^m \to \mathrm{T}$ with kernel K induces a map of the identification spaces

$$(T^m \times C(\Sigma))/\!\sim\; \to (\mathrm{T} \times C(\Sigma))/\!\sim.$$

Now, according to [10], if Σ is a complete simplicial fan, then the latter identification space is homeomorphic to the toric variety $X_\Sigma = U(\Sigma)/\mathrm{G}$. This proves (a). (Note that if Σ is a regular fan, then $\mathrm{K} \cong T^{m-n}$ and the projection $\mathcal{Z}_\Sigma \to X_\Sigma$ is a principal K-bundle.)

As for (b), there are obvious equivariant deformation retractions $U(\sigma) \to \mathcal{Z}(\sigma)$ for all $\sigma \in \Sigma$, which patch together to get the necessary map $U(\Sigma) \to \mathcal{Z}(\Sigma)$.

By comparing this result with Theorem 2.1, we see that $\mathcal{Z}(\Sigma)$ has the same properties with respect to the G-action on $U(\Sigma)$ as the set KN_S with respect to the G-action on an affine variety S. We therefore refer to $\mathcal{Z}(\Sigma)$ as the *Kempf–Ness* set of $U(\Sigma)$.

Example 3.5 Let $n = 2$ and e_1, e_2 be a basis in $N_{\mathbb{R}}$.
1. Consider a complete fan Σ having the following three 2-dimensional cones: the first is spanned by e_1 and e_2, the second spanned by e_2 and $-e_1-e_2$, and the third spanned by $-e_1-e_2$ and e_1. The simplicial complex $\mathcal{K}_\Sigma$ is a complete graph on 3 vertices (or the boundary of a triangle). We have

$$U(\Sigma) = \mathbb{C}^3 \setminus \{z\colon z_1 = z_2 = z_3 = 0\} = \mathbb{C}^3 \setminus \{0\}$$

and

$$\mathcal{Z}(\Sigma) = \mathrm{D}^2 \times \mathrm{D}^2 \times \mathrm{S}^1 \cup \mathrm{D}^2 \times \mathrm{S}^1 \times \mathrm{D}^2 \cup \mathrm{S}^1 \times \mathrm{D}^2 \times \mathrm{D}^2 = \partial((\mathrm{D}^2)^3) \cong \mathrm{S}^5.$$

The subgroup G from exact sequence (3.1) is the diagonal 1-dimensional subtorus $\mathbb{C}^*_d \subset (\mathbb{C}^*)^3$, and K is the diagonal subcircle $\mathrm{S}^1_d \subset T^3$. Therefore, we have $X_\Sigma = U(\Sigma)/\mathrm{G} = \mathcal{Z}(\Sigma)/\mathrm{K} = \mathbb{C}P^2$, the complex projective 2-plane.

2. Now consider the fan Σ consisting of three 1-dimensional cones generated by vectors e_1, e_2 and $-e_1 - e_2$. This fan is not complete, but its 1-dimensional cones span $N_{\mathbb{R}}$ as a vector space. So Theorem 3.2 applies, but Theorem 3.4 (a) does not. The simplicial complex $\mathcal{K}_\Sigma$ consists of 3 disjoint points. The space $U(\Sigma)$ is the complement to 3 coordinate lines in $\mathbb{C}^3$:

$$U(\Sigma) = \mathbb{C}^3 \setminus \{z\colon z_1 = z_2 = 0, \quad z_1 = z_3 = 0, \quad z_2 = z_3 = 0\},$$

and

$$\mathcal{Z}(\Sigma) = \mathrm{D}^2 \times \mathrm{S}^1 \times \mathrm{S}^1 \cup \mathrm{S}^1 \times \mathrm{D}^2 \times \mathrm{S}^1 \cup \mathrm{S}^1 \times \mathrm{S}^1 \times \mathrm{D}^2.$$

Both spaces are homotopy equivalent to $\mathrm{S}^3 \vee \mathrm{S}^3 \vee \mathrm{S}^3 \vee \mathrm{S}^4 \vee \mathrm{S}^4$ (see Example 8.15 of [6] and [12]). Like in the previous example, the subgroup G is the diagonal subtorus $\mathbb{C}^*_d \subset (\mathbb{C}^*)^3$. By Theorem 3.2, $X_\Sigma = U(\Sigma)/\mathrm{G}$, a quasiprojective variety obtained by removing three points from $\mathbb{C}P^2$. This in non-compact, and cannot be identified with $\mathcal{Z}(\Sigma)/\mathrm{K}$.

4. Normal fans

The next step in our study of the Kempf–Ness set for torus actions on quasiaffine varieties $U(\Sigma)$ would be to obtain an explicit description like the one given by (2.1) in the affine case. Although we do not now of such a description in general, it does exist in the particular case when Σ is the normal fan of a simple polytope.
Let $M_{\mathbb{R}} = (N_{\mathbb{R}})^*$ be the dual vector space. Assume we are given primitive vectors $a_1, \ldots, a_m \in N$ and integer numbers $b_1, \ldots, b_m \in \mathbb{Z}$, and consider the set

$$P = \{x \in M_{\mathbb{R}}\colon \langle a_i, x\rangle + b_i \geqslant 0, \quad i = 1, \ldots, m\}. \tag{4.1}$$

We further assume that P is bounded, the affine hull of P is the whole $M_{\mathbb{R}}$, and the intersection of P with every hyperplane determined by the equation $(a_i, x) + b_i = 0$ spans an affine subspace of dimension $n-1$, for $i = 1, \ldots, m$ (or, equivalently, none of the inequalities can be removed without enlarging P). This means that P is a *convex polytope* with exactly *m facets*. (In general, the set P is always convex, but it may be unbounded, not of the full dimension, or there may be redundant inequalities.) By introducing a Euclidean metric in $N_{\mathbb{R}}$ we may think of a_i as the inward pointing normal vector to the corresponding facet F_i of P, $i = 1, \ldots, m$. Given a face $Q \subset P$ we say that a_i is *normal* to Q if $Q \subset F_i$. If Q is an q-dimensional face, then the set of all its normal vectors $\{a_{i_1}, \ldots, a_{i_k}\}$ spans an $(n-q)$-dimensional cone σ_Q. The collection of cones $\{\sigma_Q : Q \text{ a face of } P\}$ is a complete fan in N, which we denote Σ_P and refer to as the *normal fan* of P. The normal fan is simplicial if and only if the polytope P is *simple*, that is, there exactly n facets meeting at each of its vertices. In this case the cones of Σ_P are generated by subsets $\{a_{i_1}, \ldots, a_{i_k}\}$ such that the intersection $F_{i_1} \cap \ldots \cap F_{i_k}$ of the corresponding facets is non-empty.

The Kempf–Ness sets (or the moment angle complexes) $\mathcal{Z}(\Sigma_P)$ corresponding to normal fans of simple polytopes admit a very transparent interpretation as *complete intersections* of *real algebraic quadrics*, as described in [7] (these complete intersections of quadrics were also studied in [5]). We give this construction below.

In the rest of this section we assume that P is a simple polytope, and therefore, Σ_P is a simplicial fan. We may specify P by a matrix inequality $A_P x + b_P \geqslant 0$, where A_P is the $m \times n$ matrix of row vectors a_i, and b_P is the column vector of scalars b_i. The linear transformation $M_{\mathbb{R}} \to \mathbb{R}^m$ defined by the matrix A_P is exactly the one obtained from the map $T^m \to \mathrm{T}$ in (3.2) by applying $\mathrm{Hom}_{\mathbb{Z}}(\,\cdot\,, \mathrm{S}^1) \otimes_{\mathbb{Z}} \mathbb{R}$. Since the points of P are specified by the constraint $A_P x + b_P \geqslant 0$, the formula $i_P(x) = A_P x + b_P$ defines an affine injection

$$i_P \colon M_{\mathbb{R}} \longrightarrow \mathbb{R}^m, \tag{4.2}$$

which embeds P into the positive cone $\mathbb{R}^m_{\geqslant} = \{y \in \mathbb{R}^m : y_i \geqslant 0\}$.

Now define the space $\mathcal{Z}_P$ by a pullback diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{\ i_Z\ } & \mathbb{C}^m \\ {\scriptstyle \varrho_P}\big\downarrow & & \big\downarrow{\scriptstyle \varrho} \\ P & \xrightarrow{\ i_P\ } & \mathbb{R}^m \end{array} \tag{4.3}$$

where $\varrho(z_1, \ldots, z_m)$ is given by $(|z_1|^2, \ldots, |z_m|^2)$. The vertical maps above are projections onto the quotients by the T^m-actions, and i_Z is a T^m-equivariant embedding.

Proposition 4.1 (a) *We have* $\mathcal{Z}_P \subset U(\Sigma_P)$.
(b) *There is a* T^m*-equivariant homeomorphism* $\mathcal{Z}_P \cong \mathcal{Z}(\Sigma_P)$.

Proof. Assume $z \in \mathcal{Z}_P \subset \mathbb{C}^m$ and let $\omega(z)$ be the set of zero coordinates of z. Since the facet F_i of P is the intersection of P with the hyperplane $(a_i, x)+b_i = 0$, the point $\varrho_P(z)$ belongs to the intersection $\cap_{i\in\omega(z)} F_i$, which is thereby non-empty. Therefore, the vectors $\{a_i : i \in \omega(z)\}$ span a cone of Σ_P. Thus, $\omega(z)$ is a g-subset and $z \in U(\Sigma_P)$, proving (a).

To prove (b) we look more closely on the construction of the identification space from the proof of Theorem 3.4 in the case when Σ is a normal fan. Then the space $C(\Sigma_P)$ may be identified with P, and $C(\sigma)$ is the face $\cap_{i\in g(\sigma)} F_i$ of P. The Kempf–Ness set $\mathcal{Z}(\Sigma_P)$ is therefore identified with

$$(T^m \times P)/\sim. \tag{4.4}$$

Now we notice that if we replace P by the positive cone $\mathbb{R}^m_{\geqslant}$ (with the obvious face structure) in the above identification space, we obtain $(T^m \times \mathbb{R}^m_{\geqslant})/\sim\, = \mathbb{C}^m$. Since the map i_P from (4.3) respects facial codimension, the pullback space $\mathcal{Z}_P$ can be also identified with (4.4), thus proving (b).

Choosing a basis for $\operatorname{coker} A_P$ we obtain a $(m-n)\times m$-matrix C so that the resulting short exact sequence

$$0 \longrightarrow M_{\mathbb{R}} \xrightarrow{A_P} \mathbb{R}^m \xrightarrow{C} \mathbb{R}^{m-n} \longrightarrow 0, \tag{4.5}$$

is the one obtained from (3.2) by applying $\operatorname{Hom}_{\mathbb{Z}}(\,\cdot\,, \mathrm{S}^1) \otimes_{\mathbb{Z}} \mathbb{R}$.

We may assume that the first n normal vectors $a_1, \ldots, a_n$ span a cone of Σ_P (equivalently, the corresponding facets of P meet at a vertex), and take these vectors as a basis of $M_{\mathbb{R}}$. In this basis, the first minor of the matrix (a_{ij}) of A_P is a unit $n \times n$-matrix, and we may take

$$C = \begin{pmatrix} -a_{n+1,1} & \ldots & -a_{n+1,n} & 1 & 0 & \ldots & 0 \\ -a_{n+2,1} & \ldots & -a_{n+2,n} & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{m,1} & \ldots & -a_{m,n} & 0 & 0 & \ldots & 1 \end{pmatrix}. \tag{4.6}$$

Then diagram (4.3) implies that i_Z embeds $\mathcal{Z}_P$ in $\mathbb{C}^m$ as the space of solutions of the $m-n$ real quadratic equations

$$\sum_{k=1}^{m} c_{jk}\left(|z_k|^2 - b_k\right) = 0, \quad \text{for } 1 \le j \le m-n \tag{4.7}$$

where $C = (c_{jk})$ is given by (4.6). This intersection of real quadrics is non-degenerate (the normal vectors are linearly independent at each point, see Lemma 3.1 in [7]), and therefore, $\mathcal{Z}_P \subset \mathbb{R}^{2m}$ is a smooth submanifold with trivial normal bundle.

5. Projective toric varieties and moment maps

In the notation of Section 2, let $f_v = (dF_v)_e \colon \mathfrak{g} \to \mathbb{R}$. It maps $\gamma \in \mathfrak{g}$ to $\langle \gamma v, v\rangle$, see (2.1). We may consider f_v as an element of the dual Lie algebra $\mathfrak{g}^*$. As G is reductive, we have $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. Since K is norm preserving, f_v vanishes on $\mathfrak{k}$, so we consider f_v as an element of $i\mathfrak{k}^* \cong \mathfrak{k}^*$. Varying $v \in V$ we get the *moment map* $\mu \colon V \to \mathfrak{k}^*$, which sends $v \in V$, $\kappa \in \mathfrak{k}$ to $\langle i\kappa v, v\rangle$. The Kempf–Ness set is the set of zeroes of μ:

$$KN = \mu^{-1}(0). \tag{5.1}$$

This description does not apply to the case of algebraic torus actions on $U(\Sigma)$ considered in the two previous sections: as is seen from simple examples below, the set $\mu^{-1}(0) = \{z \in \mathbb{C}^m \colon \langle \kappa z, z\rangle = 0 \text{ for all } \kappa \in \mathfrak{k}\}$ consist only of the origin in this case. Nevertheless, in this section we show that a description of the Kempf–Ness set $\mathcal{Z}(\Sigma)$ similar to (5.1) exists in the case when Σ is a normal fan, thereby extending the analogy with Kempf–Ness sets for affine varieties even further.

As explained in [3] or in §6.1 of [6], the toric variety X_Σ is projective exactly when Σ arises as the normal fan of a convex polytope. In fact, the set of integers $\{b_1, \dots, b_m\}$ from 4.1 determines an *ample* divisor on X_{Σ_P}, therefore providing a projective embedding. Note that the vertices of P are not necessarily lattice points in M (as they may have rational coordinates), but this can be remedied by simultaneously multiplying $b_1, \dots, b_m$ by an integer number; this corresponds to the passage from an ample divisor to a *very ample* one.

Assume now that Σ_P is a regular fan, therefore, X_{Σ_P} is a smooth projective variety. This implies that X_{Σ_P} is Kähler, and therefore, a symplectic manifold. There is the following symplectic version of the construction from Section 3.

Let (W, ω) be a symplectic manifold with a K-action that preserves the symplectic form ω. For every $\kappa \in \mathfrak{k}$ we denote by ξ_κ the corresponding

K-invariant vector field on W. The K-action is said to be *Hamiltonian* if the 1-form $\omega(\,\cdot\,,\xi_\kappa)$ is exact for every $\kappa\in\mathfrak{k}$, that is, there is a function H_κ on W such that

$$\omega(\xi,\xi_\kappa)=dH_\kappa(\xi)=\xi(H_\kappa)$$

for every vector field ξ on W. Under this assumption, the *moment map*

$$\mu\colon W\to\mathfrak{k}^*,\qquad (x,\kappa)\mapsto H_\kappa(x)$$

is defined.

Example 5.1 1. Basic example is given by $W=\mathbb{C}^m$ with the symplectic form $\omega=2\sum_{k=1}^m dx_k\wedge dy_k$, where $z_k=x_k+iy_k$. The coordinatewise action of $\mathrm{K}=T^m$ is Hamiltonian with the moment map $\mu\colon\mathbb{C}^m\to\mathbb{R}^m$ given by $\mu(z_1,\ldots,z_m)=(|z_1|^2,\ldots,|z_m|^2)$ (we identify the dual Lie algebra of T^m with $\mathbb{R}^m$).

2. Now let Σ be a simplicial fan, and K be the subgroup of T^m defined by (3.2). We can restrict the previous example to the K-action on the invariant subvariety $U(\Sigma)\subset\mathbb{C}^m$. The corresponding moment map is then defined by the composition

$$\mu_\Sigma\colon\mathbb{C}^m\longrightarrow\mathbb{R}^m\longrightarrow\mathfrak{k}^*. \tag{5.2}$$

A choice of an isomorphism $\mathfrak{k}\cong\mathbb{R}^{m-n}$ allows to identify the map $\mathbb{R}^m\to\mathfrak{k}^*$ with the linear transformation given by matrix (4.6), see (4.5).

A direct comparison with (5.1) prompts us to relate the level set $\mu_\Sigma^{-1}(0)$ of moment map (5.2) with the Kempf–Ness set $\mathcal{Z}_P$ for the G-action on $U(\Sigma_P)$. However, this analogy is not that straightforward: the set $\mu_\Sigma^{-1}(0)=\{z\in\mathbb{C}^m\colon\langle\kappa z,z\rangle=0$ for all $\kappa\in\mathfrak{k}\}$ is given by the equations $\sum_{k=1}^m c_{jk}|z_k|^2=0$, $1\le j\le m-n$, which have only zero solution. (Indeed, as the intersection of $\mathbb{R}^m_\geqslant$ with the affine n-plane $i_P(M_\mathbb{R})=A_P(M_\mathbb{R})+b_P$ is bounded, its intersection with the plane $A_P(M_\mathbb{R})$ consists only of the origin.) On the other hand, by comparing (5.2) with (4.7), we obtain

Proposition 5.2 *Let Σ_P be the normal fan of a simple polytope given by* (4.1)*, and* (5.2) *the corresponding moment map. Then the Kempf–Ness set $\mathcal{Z}(\Sigma_P)$ for the* G*-action on $U(\Sigma_P)$ is given by*

$$\mathcal{Z}(\Sigma_P)\cong\mu_{\Sigma_P}^{-1}(Cb_P).$$

In other words, the difference with the affine situation is that we have to take Cb_P instead of 0 as the value of the moment map. The reason is that Cb_P is a *regular value* of μ, unlike 0.

Changing the numbers b_i by some b'_i in (4.1) (but preserving the vectors a_i) we obtain another convex set P' determined by (4.1). Assume that P' is a polytope satisfying the additional properties listed straight after (4.1) (this is equivalent to assuming that P and P' are of the same *combinatorial type*). Then the normal fans of P and P' are the same, and the manifolds $\mathcal{Z}_P$ and $\mathcal{Z}_{P'}$ defined by (4.7) are T^m-equivariantly homeomorphic. Moreover, $Cb_{P'}$, considered as an element of $\mathfrak{k}^* = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}, \mathrm{S}^1) \otimes_{\mathbb{Z}} \mathbb{R} \cong H^2(X_{\Sigma_P}; \mathbb{R})$, belongs to the *Kähler cone* of the toric variety X_{Σ_P} (see §4 of [9]). In the case of normal fans the following version of our Theorem 3.4 (a) is known in toric geometry (cf. Theorem 4.1 of [9]):

Theorem 5.3 *Let X_Σ be a projective simplicial toric variety and assume that $c \in H^2(X_\Sigma; \mathbb{R})$ is in the Kähler cone. Then $\mu_\Sigma^{-1}(c) \subset U(\Sigma)$, and the natural map*

$$\mu_\Sigma^{-1}(c)/\mathrm{K} \to U(\Sigma)/\mathrm{G} = X_\Sigma$$

is a diffeomorphism.

This statement is the essence of the construction of smooth projective toric varieties via *symplectic reduction*. The submanifold $\mu_\Sigma^{-1}(c) \subset \mathbb{C}^m$ may fail to be symplectic because the restriction of the standard symplectic form ω on $\mathbb{C}^m$ to $\mu_\Sigma^{-1}(c)$ may be degenerate. However, the restriction of ω descends to the quotient $\mu_\Sigma^{-1}(c)/\mathrm{K}$ as a symplectic form.

Example 5.4 Let $P = \Delta^n$ be the *standard simplex* defined by $n+1$ inequalities $\langle e_i, x\rangle \geqslant 0,\ \ i = 1, \dots, n$, and $\langle(-1, \dots, -1), x\rangle + 1 \geqslant 0$ in $M_{\mathbb{R}}$ (here $e_1, \dots, e_n$ is a chosen basis which we use to identify $N_{\mathbb{R}}$ with $\mathbb{R}^n$). The cones of the corresponding normal fan Σ are generated by the proper subsets of the set of vectors $\{e_1, \dots, e_n, (-1, \dots, -1)\}$. The groups $\mathrm{G} \cong \mathbb{C}^*$ and $\mathrm{K} \cong \mathrm{S}^1$ are the diagonal subgroups in $(\mathbb{C}^*)^{n+1}$ and T^{n+1} respectively, while $U(\Sigma) = \mathbb{C}^{n+1} \setminus \{0\}$. The $(n+1) \times n$-matrix $A_P = (a_{ij})$ has $a_{ij} = \delta_{ij}$ for $1 \leqslant i, j \leqslant n$ and $a_{n+1,j} = -1$ for $1 \leqslant j \leqslant n$. The matrix C (4.6) is just one row of units. Moment map (5.2) is given by $\mu_\Sigma(z_1, \dots, z_{n+1}) = |z_1|^2 + \ldots + |z_{n+1}|^2$. Since $Cb_P = 1$, the Kempf–Ness set $\mathcal{Z}_P = \mu_\Sigma^{-1}(1)$ is the unit sphere $\mathrm{S}^{2n+1} \subset \mathbb{C}^{n+1}$, and $X_\Sigma = (\mathbb{C}^{n+1} \setminus \{0\})/\mathrm{G} = \mathrm{S}^{2n+1}/\mathrm{K}$ is the complex projective space $\mathbb{C}P^n$.

In the next section we consider a more complicated example, while here we finish with an open question.

Problem 5.5 As is known (see e.g., Ch. 5 of [6]), there are many complete regular fans Σ which cannot be realized as normal fans of convex polytopes.

The corresponding toric varieties X_Σ are not projective (although being non-singular). In this case the Kempf–Ness set $\mathcal{Z}(\Sigma)$ is still defined (see Section 3), as well as the moment map (5.2). However, the rest of the analysis of the last two sections does not apply here; in particular, we don't have a description of $\mathcal{Z}(\Sigma)$ as in (4.7). Can one still describe $\mathcal{Z}(\Sigma)$ as a complete intersection of real quadratic (or other) hypersurfaces? And does the moment map $\mu_\Sigma\colon U(\Sigma) \to \mathfrak{k}^*$ have any regular values?

6. Cohomology of Kempf–Ness sets

Here we use the results of [6] and [15] on moment-angle complexes to describe the integer cohomology rings of toric Kempf–Ness sets. As we shall see from an example below, the topology of $\mathcal{Z}(\Sigma)$ may be quite complicated even for simple fans.

Given an abstract simplicial complex $\mathcal{K}$ on the set $[m]$, the *face ring* [17] (or the *Stanley–Reisner ring*) $\mathbb{Z}[\mathcal{K}]$ is defined as the following quotient of the polynomial ring on m generators:

$$\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, \ldots, v_m]/(v_{i_1} \cdots v_{i_k} \colon \{i_1, \ldots, i_k\} \text{ is not a simplex of } \mathcal{K}).$$

We introduce a grading by setting $\deg v_i = 2, \quad i = 1, \ldots, m$. As $\mathbb{Z}[\mathcal{K}]$ may be thought of as a $\mathbb{Z}[v_1, \ldots, v_m]$-module via the projection map, the bigraded Tor-*modules* $\mathrm{Tor}^{-i,2j}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z})$ are defined, see [17]. They can be calculated, for example, using the *Koszul resolution* of the trivial $\mathbb{Z}[v_1, \ldots, v_m]$-module $\mathbb{Z}$. This also endows $\mathrm{Tor}^{*}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z})$ with a graded commutative algebra structure (the grading is by the total degree), see details in Ch. 7 of [6]. The following result is proved as Theorems 7.6–7.7 in [6], or Theorem 4.7 in [15].

Theorem 6.1 *For every simplicial fan Σ there are algebra isomorphisms*

$$H^*\big(\mathcal{Z}(\Sigma); \mathbb{Z}\big) \cong \mathrm{Tor}^{*}_{\mathbb{Z}[v_1,\ldots,v_m]}\big(\mathbb{Z}[\mathcal{K}_\Sigma], \mathbb{Z}\big) \cong H\big[\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}_\Sigma], d\big],$$

where the latter denotes the cohomology of a differential graded algebra with $\deg u_i = 1$, $\deg v_i = 2$, $du_i = v_i$, $dv_i = 0$ *for* $1 \leqslant i \leqslant m$.

Given a subset $I \subseteq [m]$, denote by $\mathcal{K}(I)$ the corresponding *full subcomplex* of $\mathcal{K}$, or the restriction of $\mathcal{K}$ to I. We also denote by $\widetilde{H}^i(\mathcal{K}(I))$ the ith reduced simplicial cohomology group of $\mathcal{K}(I)$ with integer coefficients. A theorem due to Hochster [13] expresses the Tor-modules $\mathrm{Tor}^{-i,2j}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z})$ in terms of full subcomplexes of $\mathcal{K}$, which leads to the following description of the cohomology of $\mathcal{Z}(\Sigma)$ (cf. Corollary 5.2 in [15]).

Theorem 6.2 *We have*

$$H^k\big(\mathcal{Z}(\Sigma)\big) \cong \bigoplus_{I\subseteq[m]} \widetilde{H}^{k-|I|-1}\big(\mathcal{K}_\Sigma(I)\big).$$

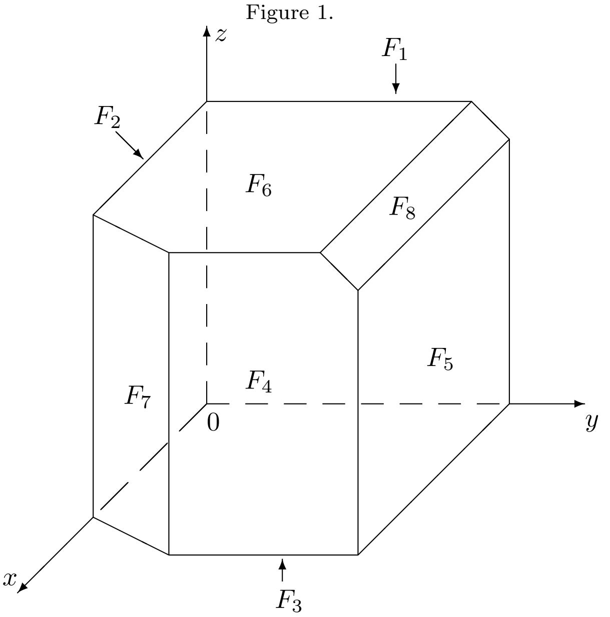
There is also a description of the product in $H^*(\mathcal{Z}(\Sigma))$ in terms of full subcomplexes of $\mathcal{K}_\Sigma$, cf. Theorem 5.1 of [15].

Example 6.3 Let P be the simple polytope obtained by cutting two non-adjacent edges off a cube in $M_\mathbb{R} \cong \mathbb{R}^3$, as shown on Fig. 1. We may specify such a polytope by 8 inequalities

$$x \geqslant 0, \quad y \geqslant 0, \quad z \geqslant 0, \quad -x+3 \geqslant 0, \quad -y+3 \geqslant 0, \quad -z+3 \geqslant 0,$$
$$-x+y+2 \geqslant 0, \quad -y-z+5 \geqslant 0,$$

and it has 8 facets $F_1, \dots, F_8$, numbered as on the picture.

Figure 1.



The 1-dimensional cones of the corresponding normal fan Σ_P are spanned by the following primitive vectors:

$$a_1 = e_1, \quad a_2 = e_2, \quad a_3 = e_3, \quad a_4 = -e_1, \quad a_5 = -e_2, \quad a_6 = -e_3,$$
$$a_7 = -e_1 + e_2, \quad a_8 = -e_2 - e_3.$$

Toric variety X_{Σ_P} is obtained by blowing up the product $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ (corresponding to the cube) at two complex 1-dimensional subvarieties $\{\infty\} \times \{0\} \times \mathbb{C}P^1$ and $\mathbb{C}P^1 \times \{\infty\} \times \{\infty\}$. Matrix (4.6) is given by

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Its transpose determines the inclusion $\mathrm{G} \hookrightarrow (\mathbb{C}^*)^8$ (or $\mathrm{K} \hookrightarrow T^8$), and we have $X_{\Sigma_P} = U(\Sigma_P)/\mathrm{G} = \mathcal{Z}(\Sigma_P)/\mathrm{K}$ by Theorem 3.4. The Kempf–Ness set $\mathcal{Z}(\Sigma_P) \cong \mathcal{Z}_P$ (4.7) is defined by 5 real quadratic equations:

$$|z_1|^2 + |z_4|^2 - 3 = 0, \quad |z_2|^2 + |z_5|^2 - 3 = 0, \quad |z_3|^2 + |z_6|^2 - 3 = 0,$$
$$|z_1|^2 - |z_2|^2 + |z_7|^2 - 2 = 0, \quad |z_2|^2 + |z_3|^2 + |z_8|^2 - 5 = 0.$$

The dual triangulation $\mathcal{K}_\Sigma$ is obtained from the boundary of an octahedron by applying two stellar subdivisions at non-adjacent edges [14]. The face ring is

$$\mathbb{Z}[\mathcal{K}_\Sigma] = \mathbb{Z}[v_1, \ldots, v_8]/(v_1v_4, v_1v_7, v_2v_4, v_2v_5, v_2v_8, v_3v_6, v_3v_8, v_5v_6, v_5v_7, v_7v_8).$$

According to Theorem 6.2, the group $H^3(\mathcal{Z}_P)$ has a generator for every pair of vertices of $\mathcal{K}_\Sigma$ not joined by an edge (equivalently, for every pair of non-adjacent facets of P). Therefore, $H^3(\mathcal{Z}_P) \cong \mathbb{Z}^{10}$, and the generators are represented by the following 3-cocycles in the differential graded algebra from Theorem 6.1:

$$u_1v_4,\ u_1v_7,\ u_2v_4,\ u_2v_5,\ u_2v_8,\ u_3v_6,\ u_3v_8,\ u_5v_6,\ u_5v_7,\ u_7v_8.$$

Using Theorem 6.2 again, we see that only reduced 0-cohomology of 3-vertex full subcomplexes of $\mathcal{K}_\Sigma$ may contribute to $H^4(\mathcal{Z}_P)$. There are two types of disconnected simplicial complexes on 3 vertices: "3 disjoint points" and "an edge and a point". $\mathcal{K}_\Sigma$ contains no full subcomplexes of the first type and 16 subcomplexes of the second type. The corresponding 4-cocycles in the differential graded algebra $\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}_\Sigma]$ are

$$u_4u_7v_1,\ u_4u_5v_2,\ u_4u_8v_2,\ u_5u_8v_2,\ u_6u_8v_3,\ u_1u_2v_4,\ u_2u_6v_5,\ u_2u_7v_5,$$
$$u_6u_7v_5,\ u_3u_5v_6,\ u_1u_5v_7,\ u_1u_8v_7,\ u_5u_8v_7,\ u_2u_3v_8,\ u_2u_7v_8,\ u_3u_7v_8.$$

Therefore, $H^4(\mathcal{Z}_P) \cong \mathbb{Z}^{16}$.

The 5th cohomology group of $\mathcal{Z}_P$ is the sum of the 1st cohomology of 3-vertex full subcomplexes of $\mathcal{K}_\Sigma$ and the reduced 0-cohomology of 4-vertex full subcomplexes. A 3-vertex full subcomplex of $\mathcal{K}_\Sigma$ may have non-zero 1st cohomology group only if the corresponding 3 facets of P form a "belt", that is, are pairwise adjacent but do not share a common vertex. As there are no such 3-facet belts in P, only reduced 0-cohomology of 4-vertex subcomplexes contributes in $H^5(\mathcal{Z}_P)$. The corresponding 5-cocycles are

$$u_1u_5u_8v_7,\ u_2u_3u_7v_8,\ u_4u_5u_8v_2,\ u_2u_6u_7v_5,\ u_2u_7u_5v_8 - u_2u_7u_8v_5$$

(note that the last cocycle cannot be represented by a monomial). Therefore, $H^5(\mathcal{Z}_P) \cong \mathbb{Z}^5$. Due to Poincaré duality, this completely determines the Betti vector $(1,0,0,10,16,5,5,16,10,0,0,1)$ of the 11-dimensional manifold $\mathcal{Z}_P$. The generators of the sixth cohomology group, $H^6(\mathcal{Z}_P) \cong \mathbb{Z}^5$, correspond to the 4-facet belts in P, and the corresponding 6-cocycles are

$$u_2u_3v_4v_6,\ u_1u_5v_4v_6,\ u_1u_3v_6v_7,\ u_1u_3v_4v_8,\ u_1u_3v_4v_6.$$

These are the Poincaré duals to the 5-cocycles. The fundamental class of $\mathcal{Z}_P$ is represented (up to a sign) by the cocycle $u_4u_5u_6u_7u_8v_1v_2v_3$, or by any cocycle of the form $u_{\sigma(4)}u_{\sigma(5)}u_{\sigma(6)}u_{\sigma(7)}u_{\sigma(8)}v_{\sigma(1)}v_{\sigma(2)}v_{\sigma(3)}$ where $\sigma \in S_8$ is a transposition such that the facets $F_{\sigma(1)}$, $F_{\sigma(2)}$ and $F_{\sigma(3)}$ share a common vertex.

The multiplicative structure in $H^*(\mathcal{Z}_P)$ can be easily retrieved from this description. For example, we have identities

$$[u_1v_4]\cdot[u_1v_7] = 0, \quad [u_1v_7]\cdot[u_2v_4] = 0, \quad [u_1v_4]\cdot[u_3v_6] = [u_1u_3v_4v_6],$$
$$[u_2v_4]\cdot[u_3v_6]\cdot[u_1u_5u_8v_7] = [u_1u_2u_3u_5u_8v_4v_6v_7], \quad \text{etc.}$$

Yet another interesting feature of the manifold $\mathcal{Z}_P$ of this Example is the existence of non-trivial Massey products in $H^*(\mathcal{Z}_P)$ [1]. Consider 3 cocycles $a = u_1v_4$, $b = u_2v_5$, $c = u_3v_6$ representing cohomology classes $\alpha,\beta,\gamma \in H^3(\mathcal{Z}_P)$. Since $\alpha\beta = 0$ and $\beta\gamma = 0$, a triple Massey product $\langle\alpha,\beta,\gamma\rangle$ is defined. It consists of the cohomology classes in $H^8(\mathcal{Z}_P)$ represented by the cocycles of the form $af + ec$ for all choices of e and f such that $ab = de$ and $bc = df$ (as there are may be many choices of e and f, the Massey product is a multivalued operation in general). The Massey product is said to be *trivial* if it contains zero. In our case we may take $e = u_1u_2u_5v_4$ and $f = 0$, so $\langle\alpha,\beta,\gamma\rangle$ contains a non-zero cohomology class $[u_1u_2u_5u_3v_4v_6] \in H^8(\mathcal{Z}_P)$. Moreover, $\langle\alpha,\beta,\gamma\rangle$ is non-trivial, cf. Example 5.7 in [15]. This implies that $\mathcal{Z}_P$ is a *non-formal* manifold. A detailed study of Massey products in the cohomology of moment-angle complexes is undertaken in [11].

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