SPECTRAL ESTIMATES FOR DIFFERENTIAL FORMS *

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In this expository paper we review two recent estimates on the first eigenvalue of the Laplacian acting on differential forms. We start by recalling the classical facts on harmonic forms on closed manifolds; then we introduce the absolute and relative boundary conditions and report on recent sharp estimates of the first eigenvalue of convex Euclidean domains. Finally we give upper bounds for the first Hodge eigenvalue of isometric immersions in the Euclidean space and in the canonical sphere, discussing in particular minimal spherical immersions. Our purpose is not to give an up-to-date, complete report on the research in this field, but is rather an attempt to explain the geometric motivations of some problems which are sometimes regarded as technical.

Complete proofs will appear elsewhere and can be found in the references.

1. Review of some classical facts

1.1. *The Hodge Laplacian*

Let M be a compact, connected, oriented Riemannian manifold without boundary and Δ the Laplace operator acting on functions. The spectrum of the Laplacian is an important invariant of the manifold; as M is compact, it is an increasing sequence of non-negative numbers

$$0 = \lambda_0(M) < \lambda_1(M) \leq \cdots \leq \lambda_k(M) \leq \ldots$$

and $\lambda_k(M) \to \infty$ as $k \to \infty$. Clearly any constant function is an eigenfunction of Δ, associated to the eigenvalue 0; the first positive eigenvalue is characterized by the variational property

$$\lambda_1(M) = \inf\left\{\frac{\int_M \|\nabla f\|^2}{\int_M f^2} : f \in C^\infty(M) \setminus \{0\}, \int_M f = 0\right\},$$

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and in general the eigenvalues (resp. eigenfunctions) of the Laplacian are the critical values (resp. critical points) of the quadratic form (*energy functional*)

$$Q(f) = \int_M \|df\|^2.$$

Let us now consider the extension of Δ acting on differential p-forms, denoted by Δ_p (or simply Δ) and often called the Hodge Laplacian of the (Riemannian) manifold M:

$$\Delta_p \omega = d\delta\omega + \delta d\omega,$$

where d is the exterior derivative and δ is the co-differential, i.e. the adjoint of d with respect to the L^2-scalar product of forms defined by the metric.

The Hodge Laplacian is then associated to the quadratic form:

$$Q(\omega) = \int_M \|d\omega\|^2 + \|\delta\omega\|^2.$$

It is self-adjoint and elliptic, hence it admits a discrete sequence of eigenvalues. The multiplicity of the zero eigenvalue has in fact an important topological meaning.

1.2. *Harmonic forms and topology*

A form is *harmonic* if it is in the kernel of the Hodge Laplacian: $\Delta\omega = 0$. The main motivation for considering this operator is that the dimension of the vector space of harmonic p-forms, denoted $\mathcal{H}^p(M)$, is a *topological* invariant, and in fact equals the p-th de Rham Betti number of M. This is a consequence of the *Hodge decomposition* theorem (see for example [19]) :

Theorem 1.1 *The space of differential p-forms splits as an orthogonal direct sum*

$$\Lambda^p(M) = \mathcal{H}^p(M) \oplus d(\Lambda^{p-1}(M)) \oplus \delta(\Lambda^{p+1}(M)).$$

In particular, $\mathcal{H}^p(M)$ is isomorphic to the p-th de Rham cohomology space of M, and each de Rham cohomology class has a unique harmonic representative.

It follows that analytic estimates on the Hodge Laplacian and curvature assumptions have interesting topological consequences; this idea is at the basis of the *Bochner method*.

1.3. *The Bochner method*

The classical Bochner formula relates the Laplacian of a form to its energy and to the curvature of the manifold. For a 1-form ω, it reads:

$$\langle \Delta\omega, \omega\rangle = \|\nabla\omega\|^2 + \frac{1}{2}\,\Delta(\|\omega\|^2) + Ric(\omega,\omega), \tag{1.1}$$

where Ric is the Ricci curvature of M, viewed as a quadratic form on $\Lambda^1(M)$. If ω is a harmonic 1-form, integrating on the manifold one gets:

$$0 = \int_M \|\nabla\omega\|^2 + \int_M Ric(\omega,\omega).$$

With this at hand the following consequence, first proved by Bochner, is now easy to prove:

Theorem 1.2 *Let M be an n-dimensional manifold without boundary and $b_1(M)$ its first Betti number. Then:*

- *(i) If the Ricci curvature is non-negative, then any harmonic 1-form is parallel and $b_1(M) \le n$.*
- *(ii) If the Ricci curvature is non-negative, and positive somewhere, then $b_1(M) = 0$.*
- *(iii) If the Ricci curvature is non-negative and $b_1(M) = n$ then M is isometric to a flat torus.*

(The proof of the last statement requires an additional argument).

Here is another application to Killing fields. The starting point is to observe that, if ω is the dual 1-form of a Killing field, then $\nabla\omega$ is skew symmetric (in particular, ω is co-closed) and $\Delta\omega = 2Ric(\omega)$ (now Ric is viewed as an endomorphism of $\Lambda^1(M)$): the proof is easy by straightforward manipulation and the Bochner formula. Then one has easily:

Theorem 1.3 *Let M be a compact, oriented manifold with non-positive Ricci curvature. Then:*

- *(i) Every Killing field is parallel.*
- *(ii) If in addition $Ric < 0$ at some point, then M has no non-trivial Killing fields, and in particular its isometry group is finite.*

For p-forms the Bochner (or Bochner-Weitzenböck) formula reads:

$$\langle \Delta\omega, \omega\rangle = \|\nabla\omega\|^2 + \frac{1}{2}\,\Delta(\|\omega\|^2) + W_p(\omega,\omega) \tag{1.2}$$

where now the curvature term $W_p(\omega, \omega)$ is more complicated to handle (of course we have seen that $W_1 = Ric$). However, an estimate by Gallot and Meyer in [6] shows that

$$W_p(\omega, \omega) \geq p(n-p)\gamma \|\omega\|^2, \tag{1.3}$$

where γ is a lower bound of the eigenvalues of the curvature operator of M. Recall that the curvature operator is the self-adjoint endomorphism $\mathcal{R}$ of $\Lambda^2(TM)$ (endowed with its canonical inner product), defined by:

$$\langle \mathcal{R}(X \wedge Y), Z \wedge W \rangle = R(X, Y, Z, W),$$

where R is the Riemann tensor and X, Y, Z, W are tangent vectors. If γ is a lower bound of the eigenvalues of the curvature operator, then all sectional curvatures are also bounded below by γ (however, the converse is not true).

1.4. *The Gallot-Meyer eigenvalue estimate*

Having discussed the kernel of the Hodge Laplacian, we now turn our attention to its first *positive* eigenvalue, which we will denote by the symbol:

$$\mu_1^{[p]}(M).$$

Note that the superscript $[p]$ refers to the degree of the eigenform involved.

The Gallot-Meyer estimate (1.3) and the Bochner formula are the main tools in the proof of the following lower bound, also proved in [6]:

Theorem 1.4 *Let M^n be a closed manifold with curvature operator having eigenvalues bounded below by $\gamma > 0$. Then, for all $p = 1, \ldots, n-1$:*

$$\mu_1^{[p]}(M) \geq c(n,p)\gamma,$$

where $c(n,p) = \min\{p(n-p+1), (p+1)(n-p)\}$. Equality holds for the canonical sphere.

The inequality $\mu_1^{[0]}(M) \geq n\gamma$ under the assumption $Ric \geq (n-1)\gamma$ had already been proved by Lichnerowicz.

1.5. *Boundary conditions*

We now assume that M has a *smooth boundary* ∂M, with inner unit normal vector denoted by N. We consider the *absolute* eigenvalue problem for the Hodge Laplacian:

$$\begin{cases} \Delta\omega = \mu\omega \\ i_N\omega = 0, i_N d\omega = 0 \quad \text{on} \quad \partial M. \end{cases} \tag{1.4}$$

Here $i_N\omega$ denotes interior multiplication of ω by the vector N. Let us say that the form ω is *tangential* if $i_N\omega = 0$ on the boundary (i.e. it vanishes on vectors which are normal to the boundary). For the absolute boundary conditions we then require that both ω and its exterior derivative $d\omega$ are tangential. For a form of degree 0, that is, for a function f, the absolute boundary conditions reduce to $\dfrac{\partial f}{\partial N} = 0$ on ∂M. Hence:

Remark 1.1 The absolute boundary conditions for forms generalize the Neumann boundary conditions for functions.

The dual eigenvalue problem is the *relative* one:

$$\begin{cases} \Delta\omega = \lambda\omega \\ J^*\omega = 0, J^*\delta\omega = 0 \quad \text{on} \quad \partial M. \end{cases} \tag{1.5}$$

where J^* denotes restriction of a form to the boundary. For a function, one asks $f = 0$ on the boundary. Hence:

Remark 1.2 The relative boundary conditions for forms generalize the Dirichlet boundary conditions for functions.

For example, let us see the boundary conditions for 1-forms on the cylinder $M = [0,1] \times \mathbb{S}^1$. In the coordinates (r,θ), where $r \in [0,1]$ and θ is the angular coordinate on the unit circle, a 1-forms is expressed as follows:

$$\omega = \omega_1(r,\theta)\, dr + \omega_2(r,\theta)\, d\theta.$$

Then, for the absolute boundary conditions, one requires that the functions ω_1 and $\dfrac{\partial\omega_1}{\partial\theta} - \dfrac{\partial\omega_2}{\partial r}$ vanish identically on ∂M, hence at all points $(0,\theta)$ and $(1,\theta)$, with $\theta \in [0,2\pi)$. For the relative boundary conditions, the same vanishing must hold for the functions ω_2 and $\dfrac{\partial\omega_1}{\partial r} + \dfrac{\partial\omega_2}{\partial\theta}$.

The Hodge decomposition theorem for manifolds with boundary (see for example [18] or [19]) has the following consequence, which motivates the names given to the above boundary conditions:

Theorem 1.5 *The vector space of harmonic p-forms satisfying the absolute (resp. relative) conditions is isomorphic to the p-th absolute (resp. relative) de Rham cohomology space of the pair $(M, \partial M)$.*

Let us now denote by $\mu_1^{[p]}(M)$ the first *positive* eigenvalue of the Hodge Laplacian for the absolute conditions, and by $\lambda_1^{[p]}(M)$ the first positive eigenvalue for the relative one. The Hodge $\star$ operator exchanges the two boundary conditions, hence

$$\mu_1^{[p]} = \lambda_1^{[n-p]}.$$

We have the min-max principle, characterizing these eigenvalues:

$$\mu_1^{[p]}(M) \quad \text{is the infimum of the quotient} \quad \frac{\int_M \|d\omega\|^2 + \|\delta\omega\|^2}{\int_M \|\omega\|^2}, \tag{1.6}$$

the infimum being taken over all non-vanishing p-forms ω which are *tangential* and L^2-orthogonal to the subspace $\mathcal{H}_A^{[p]}(M)$ of all harmonic forms satisfying the absolute boundary conditions. Note that in the min-max principle above we only ask that $i_N\omega = 0$: in fact, any eigenform realizing the infimum automatically satisfies $i_N d\omega = 0$.

As the Laplacian Δ commutes with both d and δ, it leaves the subspace of closed (resp. co-closed) forms invariant, hence the eigenvalue problem splits naturally and one has:

$$\mu_1^{[p]} = \min\{\mu_1^{[p]'}, \mu_1^{[p]''}\},$$

where $\mu_1^{[p]'}$ (resp. $\mu_1^{[p]''}$) is the first positive eigenvalue of Δ when acting on closed (resp. co-closed) p-forms. Finally, differentiating eigenforms gives $\mu_1^{[p]''} = \mu_1^{[p+1]'}$.

We now give a geometric interpretation of the above boundary conditions, the eigenvalues, and the variational principles for vector fields in the three dimensional Euclidean space $\mathbb{R}^3$.

1.6. *Vector fields in 3-space*

Let Ω be a bounded domain (with smooth boundary) in $\mathbb{R}^3$. Given a 1-form ω on Ω, we shall consider its dual vector field W_ω defined by:

$$\langle W_\omega, X\rangle = \omega(X),$$

for all vector fields X on Ω. The condition $i_N\omega = 0$ on the boundary then means that W_ω is indeed tangent to the boundary at any point of it. Now, the 1-form $\star d\omega$ corresponds under duality to the vector field

$$\operatorname{curl} W_\omega = \nabla \times W_\omega;$$

as the Hodge star operator intertwines the operators i_N and J^* one easily verifies that $i_N d\omega = 0$ if and only if $\operatorname{curl} W_\omega$ is normal to the boundary.

Therefore, the *absolute boundary conditions* for a vector field W are:

$$W \text{ tangent to } \partial\Omega \text{ and curl}W \text{ normal to } \partial\Omega.$$

Similarly, the *relative conditions* are:

$$W \text{ normal to } \partial\Omega \text{ and div}W = 0 \text{ on } \partial\Omega.$$

The Laplacian on vector fields is:

$$\Delta W = -\mathrm{grad}\,\mathrm{div}W - \mathrm{curl}^2 W.$$

Since the $\star$ operator is an isometry, one has $\| \star d\omega\|^2 = \|d\omega\|^2$, and the min-max principle gives:

$$\mu_1^{[1]}(\Omega) \quad \text{is the infimum of the quotient} \quad \frac{\int_\Omega \|\mathrm{curl}W\|^2 + |\mathrm{div}W|^2}{\int_\Omega \|W\|^2}, \quad (1.7)$$

taken over all non-vanishing vector fields W which are tangent to the boundary and L^2-orthogonal to the subspace $\mathcal{H}^1_A(\Omega)$ of all harmonic vector fields satisfying the absolute boundary conditions (this space is known to have dimension equal to the genus of $\partial\Omega$).

Similarly,

$$\mu_1^{[1]'}(\Omega) = \inf_W \frac{\int_\Omega |\mathrm{div}W|^2}{\int_\Omega \|W\|^2}, \quad (1.8)$$

where W is non-vanishing, curl-free, tangent to the boundary and L^2-orthogonal to $\mathcal{H}^1_A(\Omega)$; while

$$\mu_1^{[1]''}(\Omega) = \inf_W \frac{\int_\Omega \|\mathrm{curl}W\|^2}{\int_\Omega \|W\|^2}, \quad (1.9)$$

where now W is non-vanishing, divergence-free, tangent to the boundary and L^2-orthogonal to $\mathcal{H}^1_A(\Omega)$.

In the next section we will give sharp upper and lower estimates of $\mu_1^{[p]}$ when Ω is a convex body in $\mathbb{R}^n$. For a nice exposition of the Hodge decomposition for vector fields in 3-space, and related problems, we refer to [2].

2. Estimates for convex Euclidean domains

In this section Ω is a convex body in $\mathbb{R}^n$. We discuss here the behavior of the eigenvalues $\mu_1^{[p]}$ and see how they depend on the geometry of Ω. Proofs will appear in [16].

It should be remarked that estimating the eigenvalues of the Laplacian on forms is usually harder than estimating the eigenvalues for functions, because many of the tools valid for functions (like isoperimetry, symmetrization, etc.) do not extend to differential forms; for example, the classical Faber-Krahn inequality, valid for the first Dirichlet eigenvalue on functions, does not hold for p-forms, when $p \geq 1$. It is also unclear whether a Cheeger-type inequality can hold for forms. The spectral geometry of the Hodge Laplacian looks more complicated (and often more interesting) than that of the Laplacian on functions.

We restrict to convex domains because they have strong properties; counterexamples show that it is in general impossible to extend the type of the bounds given below to general domains.

Classically, the first eigenvalue on functions for the Dirichlet boundary conditions, that is

$$\lambda_1^{[0]} = \mu_1^{[n]},$$

is called the *fundamental tone* for the *fixed problem* on Ω; it is in fact the lowest sound produced by Ω, seen as a vibrating drum which is fixed on the boundary.

If the boundary is free to vibrate as well, then its lowest frequency (fundamental tone) is its first positive Neumann eigenvalue

$$\mu_1^{[0]}.$$

By abuse of language we will call

$$\mu_1^{[p]},$$

for $p = 0, \dots, n$, the *fundamental p-tone of* Ω. The problem we address in this section is the following:

Knowing all fundamental p-tones, that is, the eigenvalues $\mu_1^{[0]}, \mu_1^{[1]}, \dots, \mu_1^{[n]}$, what can we say about the geometry of the (convex) domain Ω?

2.1. *Known estimates for functions*

We start with $\mu_1^{[0]}$. One has:

$$\frac{\pi^2}{\operatorname{diam}(\Omega)^2} \leq \mu_1^{[0]}(\Omega) \leq \frac{n\pi^2}{\operatorname{diam}(\Omega)^2}, \tag{2.1}$$

The lower bound is due to Payne and Weinberger [14] while the upper bound follows easily from more general estimates found in [4]. The two bounds express the fact that $\mu_1^{[0]}$ is *equivalent* to the squared inverse of the diameter: in particular, it is large if the diameter is small, and viceversa it is small if the diameter is large.

Remark 2.1 In what follows, we will say that the eigenvalue $\mu_1^{[p]}(\Omega)$ is *equivalent* to a certain invariant $I(\Omega)$ if the quotient $\dfrac{\mu_1^{[p]}(\Omega)}{I(\Omega)}$ is bounded above and below by two positive constants which do not depend on Ω but only on the degree p and the dimension n. In that case, $\mu_1^{[p]}(\Omega) \to 0$ (resp. $\mu_1^{[p]}(\Omega) \to \infty$) if and only if $I(\Omega) \to 0$ (resp. $I(\Omega) \to \infty$).

Regarding $\mu_1^{[n]} = \lambda_1^{[0]}$ one has:

$$\frac{\pi^2}{4R(\Omega)^2} \leq \mu_1^{[n]}(\Omega) \leq \frac{\mu_1^{[n]}(B^n)}{R(\Omega)^2}, \tag{2.2}$$

where $R(\Omega)$ is now the inner radius of Ω (the radius of a largest ball sitting inside the domain), and B^n is the unit ball. The lower bound is due to Li and Yau [13] (and is valid, in greater generality, when the boundary is mean convex and the Ricci curvature is non-negative), while the upper bound follows immediately from the monotonicity property of the first Dirichlet eigenvalue. Hence, the eigenvalue $\mu_1^{[n]}(\Omega)$ is equivalent to the squared inverse of the inner radius of the domain.

Let us now turn to the eigenvalue $\mu_1^{[p]}(\Omega)$.

It has been proved by Guerini and Savo [9] that $\mu_1^{[p]''} \geq \mu_1^{[p]'}$ for all p, hence the sequence of fundamental tones is always non-decreasing, and actually

$$\mu_1^{[0]} = \mu_1^{[1]} \leq \mu_1^{[2]} \cdots \leq \mu_1^{[n]}. \tag{2.3}$$

Note the equality at the first step; it follows because $\mu_1^{[1]''} \geq \mu_1^{[1]'}$ and $\mu_1^{[0]} = \mu_1^{[1]'}$ (obtained by differentiating the first Neumann eigenfunction). Hence, for a convex domain the significant fundamental tones are exactly n, that is $\mu_1^{[1]}, \mu_1^{[2]}, \dots, \mu_1^{[n]}$. As an immediate corollary of (2.1)-(2.3) we have that, for all p:

$$\frac{\pi^2}{\mathrm{diam}(\Omega)^2} \leq \mu_1^{[p]}(\Omega) \leq \frac{\mu_1^{[n]}(B^n)}{R(\Omega)^2}.$$

However, we want to give a more precise estimate, and find a geometric invariant which is equivalent to $\mu_1^{[p]}$. This invariant is related to the John ellipsoid of Ω.

2.2. *The John ellipsoid*

In 1948, F. John proved the following result (see [12] or also [1]):

Theorem 2.1 *Let Ω be a convex body in $\mathbb{R}^n$. Then there is a unique ellipsoid $\mathcal{E}$ of maximal volume contained in Ω. Moreover, one has:*

$$\Omega \subset n\mathcal{E}.$$

(Here the homothety is taken with respect to the center of $\mathcal{E}$).

$\mathcal{E}$ is called the *John ellipsoid* of Ω. In some sense, it is the included ellipsoid which best approximates Ω. The inclusion expresses the fact that, if

$$\gamma(\Omega) = \inf\{t \geq 1 : \Omega \subset t\mathcal{E}\},$$

then $\gamma(\Omega) \leq n$ for all Ω. The invariant $\gamma(\Omega)$ is also called the *Banach-Mazur distance of Ω from the unit ball.*

2.3. *The main estimate*

Let then Ω be a convex body and $\mathcal{E}$ the John ellipsoid of Ω. Let $D_k(\mathcal{E})$ be the k-th principal axis of $\mathcal{E}$, and assume the ordering:

$$D_1(\mathcal{E}) \geq D_2(\mathcal{E}) \geq \cdots \geq D_n(\mathcal{E}).$$

By abuse of language, we will call $D_k(\mathcal{E})$ the *k-th principal axis of* Ω. By John's theorem, $D_1(\mathcal{E})$ is equivalent to the diameter and $D_n(\mathcal{E})$ is equivalent to the inner radius, because:

$$D_1(\mathcal{E}) \leq \operatorname{diam}(\Omega) \leq nD_1(\mathcal{E}) \quad \text{and} \quad D_n(\mathcal{E}) \leq 2R(\Omega) \leq nD_n(\mathcal{E}).$$

With that in mind, we observe that the classical estimates for functions can be stated as follows (recall that $\mu_1^{[0]} = \mu_1^{[1]}$):
$\mu_1^{[1]}$ is equivalent to $D_1(\mathcal{E})^{-2}$ and $\mu_1^{[n]}$ is equivalent to $D_n(\mathcal{E})^{-2}$.

Our main result is that this fact holds in all degrees:
For all degrees p the eigenvalue $\mu_1^{[p]}$ is equivalent to $D_p(\mathcal{E})^{-2}$.

Here is the precise statement with explicit constants (this is the main result in [16]).

Theorem 2.2 *Let Ω be a convex body in $\mathbb{R}^n$, $n \geq 3$, and $\mathcal{E}$ the John ellipsoid of Ω, with principal axes: $D_1(\mathcal{E}) \geq D_2(\mathcal{E}) \geq \cdots \geq D_n(\mathcal{E})$. Then, for all $p = 2, \dots, n-1$ one has:*

$$\frac{a_{n,p}}{D_p(\mathcal{E})^2} < \mu_1^{[p]}(\Omega) < \frac{a'_{n,p}}{D_p(\mathcal{E})^2},$$

where

$$a_{n,p} = \frac{1}{n^2 \cdot \binom{n}{p-1}}, \quad a'_{n,p} = p(n+2)n^n.$$

The strict inequality expresses the fact that the numerical constants are not sharp. We now observe some consequences.

2.4. *Collapsing*

Let us say that the sequence of convex domains $\Omega_\alpha, \alpha > 0$ in $\mathbb{R}^n$ collapses to $\mathbb{R}^m$ (here $m < n$) if, for large α, Ω_α is contained in an arbitrarily small tubular neighborhood of an m-dimensional subspace of $\mathbb{R}^n$. By John's theorem, it is clear that, if $\mathcal{E}_\alpha$ is the John ellipsoid of Ω_α, then:
Ω_α *collapses to* $\mathbb{R}^m$ *iff, as* $\alpha \to \infty$, $D_p(\mathcal{E}_\alpha) \to 0$ *for all* $p \geq m+1$.
Therefore, by the theorem,
Ω_α *collapses on* $\mathbb{R}^m$ *iff, as* $\alpha \to \infty$, $\mu_1^{[p]}(\Omega_\alpha) \to \infty$ *for all* $p \geq m+1$.

In particular, one can detect collapsing, and the dimension of the subspace on which it takes place, just by counting the number of fundamental tones which diverge to infinity.

2.5. *Spectral geometry of convex domains*

The last statement is typical in inverse spectral geometry: recover geometric properties of a manifold by the sole knowledge of its spectrum (for functions, or forms).

Let us recall that the spectrum of a manifold (say, for the Laplacian on functions and the Dirichlet boundary conditions) determines many important invariants, like the dimension, the volume, the volume of the boundary and, more generally, the whole set of heat invariants (i.e. the coefficients of the asymptotic expansion of the heat trace). So, having a *perfect* ear (that is, being able to detect exactly the infinitely many characteristic frequencies of the manifold) we can measure the above invariants with *absolute* precision.

Our point of view here is different: if we have only a *rough* ear (by that we mean: we can only detect the n fundamental tones of Ω) then we can *roughly* determine the John ellipsoid of Ω, where rough now means: up to (explicit) constants depending only on the degree and the dimension. In that sense, we can *roughly hear* the John ellipsoid of Ω.

In fact, the main estimate can be restated as follows: if $D_p(\mathcal{E})$ is, as usual, the p-th longest principal axis of the John ellipsoid of Ω, then

$$\sqrt{\frac{a_{n,p}}{\mu_1^{[p]}(\Omega)}} \leq D_p(\mathcal{E}) \leq \sqrt{\frac{a'_{n,p}}{\mu_1^{[p]}(\Omega)}}, \tag{2.4}$$

where the constants are given in the main theorem. (One could also say that $D_p(\mathcal{E})$ is equivalent to the p-th fundamental *wavelength* $1/\sqrt{\mu_1^{[p]}}$). By again using John's theorem, we can see that Ω lies between two suitable ellipsoids depending only on the fundamental tones:

$$\mathcal{E}^-_{spec} \subseteq \Omega \subseteq n\mathcal{E}^+_{spec}, \tag{2.5}$$

where, for $p = 1, \ldots, n$: $\mathcal{E}^-_{spec}$ is the ellipsoid of principal axes: $D_p(\mathcal{E}^-_{spec}) = \sqrt{\frac{a_{n,p}}{\mu_1^{[p]}(\Omega)}}$ and $\mathcal{E}^+_{spec}$ is the ellipsoid of principal axes: $D_p(\mathcal{E}^+_{spec}) = \sqrt{\frac{a'_{n,p}}{\mu_1^{[p]}(\Omega)}}$.

2.6. *Volume of cross-sections*

Our results imply that we can roughly hear also the maximal volume of a p-dimensional cross-section of Ω. Precisely, let:

$$\mathrm{vol}^{[p]}(\Omega) = \sup\{\mathrm{vol}(\Sigma) : \Sigma = \pi \cap \Omega,\ \pi \text{ is a p-dimensional plane}\}.$$

Clearly $\mathrm{vol}^{[1]}$ is the diameter of the domain, and $\mathrm{vol}^{[n]}$ is its usual $n-$dimensional volume. Now, for an ellipsoid $\mathcal{E}$ the above invariant is a constant multiple of the product of the longest p principal axes:

$$\mathrm{vol}^{[p]}(\mathcal{E}) = 2^{-p}\mathrm{vol}(B^p) D_1(\mathcal{E}) \cdots D_n(\mathcal{E}).$$

Since the invariant $\mathrm{vol}^{[p]}$ is monotonic with respect to inclusion, we get, by our main estimate and John's theorem:

$$\frac{c_{n,p}}{\sqrt{\mu_1^{[1]} \cdots \mu_1^{[p]}}} \leq \mathrm{vol}^{[p]}(\Omega) \leq \frac{c'_{n,p}}{\sqrt{\mu_1^{[1]} \cdots \mu_1^{[p]}}}, \tag{2.6}$$

for explicit constants $c_{n,p}, c'_{n,p}$ not depending an the domain, but only on the degree and the dimension.

As the sequence of fundamental p-tones is non-decreasing, one also gets the weaker inequality:

$$\mu_1^{[p]}(\Omega) \geq \left(\frac{c_{n,p}}{\mathrm{vol}^{[p]}(\Omega)}\right)^{2/p}.$$

2.7. *Vector fields in 3-space*

Let Ω be a convex domain in the three dimensional Euclidean space, and $\mathcal{E}$ its John ellipsoid with principal axes $D_1(\mathcal{E}) \geq D_2(\mathcal{E}) \geq D_3(\mathcal{E})$. Recall the variational characterization of the Hodge eigenvalues in terms of vector fields, given in the previous section. As $\mu_1^{[1]} = \mu_1^{[0]}$ one has from (1.7) and the Payne-Weinberger inequality (2.1):

$$\int_\Omega \|\mathrm{curl} W\|^2 + |\mathrm{div} W|^2 \geq \frac{\pi^2}{\mathrm{diam}(\Omega)^2} \int_\Omega \|W\|^2, \tag{2.7}$$

for all vector fields W which are tangent to the boundary. (Note that we could have equivalently used $D_1(\mathcal{E})$ in the lower bound).

On the other hand, if W is divergence-free and tangent to the boundary, then, as $\mu_1^{[1]''} = \mu_1^{[2]}$ one has, by the main theorem applied for $p = 2$:

$$\int_\Omega \|\mathrm{curl} W\|^2 \geq \frac{4}{27 D_2(\mathcal{E})^2} \int_\Omega \|W\|^2. \tag{2.8}$$

which is better than (2.7) because D_2 might be much smaller than the diameter. In fact, when Ω collapses onto a segment (so that both D_2 and D_3 tend to zero) the quotient $\int_\Omega \|\mathrm{curl} W\|^2 / \int_\Omega \|W\|^2$ will become infinite, no matter how big is D_1, hence the diameter.

2.8. *Polarization of eigenforms*

The estimates needed to prove the main theorem also give some information on the behavior of eigenforms under collapsing. We limit ourselves to the three dimensional example above, and assume that the convex domain Ω collapses onto a 2-plane; this happens when $D_3 \to 0$ and D_1, D_2 stay bounded below by a fixed positive constant. By the theorem, $\mu_1^{[3]}$ tends to infinity and $\mu_1^{[1]}, \mu_1^{[2]}$ stay bounded above. Let W be an eigen-vector field associated to $\mu_1^{[1]''} = \mu_1^{[2]}$. It can be shown that, if e_3 is a unit vector in the direction of D_3 (which is the direction of collapsing), then:

$$\frac{\int_\Omega \langle W, e_3 \rangle^2}{\int_\Omega \|W\|^2} \quad \text{tends to zero,}$$

showing that W tend to polarize (in the L^2-sense) along the plane orthogonal to the direction of collapsing. The polarization can be estimated by explicit constants.

3. Estimates for submanifolds

It is of interest, in spectral geometry, to relate the eigenvalues of the Laplacian to the extrinsic geometry of an isometric immersion. A classical result in this regard was given by Reilly in 1977 (see [15]): if M^n is a compact manifold without boundary and $M^n \longrightarrow \mathbb{R}^{m+n}$ is an isometric immersion, then

$$\mu_1^{[0]}(M) \leq \frac{n}{\mathrm{vol}(M)} \int_M \|H\|^2, \tag{3.1}$$

where $\mu_1^{[0]}(M)$ is the first positive eigenvalue of the Laplacian on functions and H is the mean curvature vector of the immersion. The equality holds for the canonical immersion $\mathbb{S}^n \longrightarrow \mathbb{R}^{n+1}$.

In our paper [17] we extended Reilly's estimate to the Laplacian on forms and we proved the following upper bound.

Theorem 3.1 *Let $M^n \longrightarrow \mathbb{R}^{m+n}$ be an isometric immersion. For all $p = 1, \ldots, n$ one has:*

$$\mu_1^{[p]'}(M) \leq \frac{1}{vol(M)} \int_M \left[pn\|H\|^2 - \frac{p(p-1)}{n(n-1)} scal \right],$$

where scal is the scalar curvature of M^n. The equality holds for the canonical immersion $\mathbb{S}^n \longrightarrow \mathbb{R}^{n+1}$.

Here $\mu_1^{[p]'}(M)$ denotes the first positive eigenvalue of the Laplacian restricted to the subspace of closed p-forms; then $\mu_1^{[p]}(M) \leq \mu_1^{[p]'}(M)$. The Reilly estimate (3.1) is recovered by setting $p = 1$, because $\mu_1^{[1]'} = \mu_1^{[0]}$.

The next estimate is obtained by composing an isometric immersion $M^n \longrightarrow \mathbb{S}^{m+n}$ with the canonical immersion $\mathbb{S}^{m+n} \longrightarrow \mathbb{R}^{m+n+1}$:

Corollary 3.1 *Let $M^n \longrightarrow \mathbb{S}^{m+n}$ be an isometric immersion. Then:*

$$\mu_1^{[p]'}(M) \leq pn + \frac{1}{vol(M)} \int_M \left[pn\|H\|^2 - \frac{p(p-1)}{n(n-1)} scal \right].$$

3.1. *Minimal immersions into a sphere.*

It is interesting to observe that, if M^n is minimally immersed into $\mathbb{S}^{m+n}$, then the corollary gives:

$$\mu_1^{[p]'}(M) \leq pn - \frac{p(p-1)}{n(n-1)} \cdot \frac{1}{\mathrm{vol}(M)} \int_M scal.$$

The question is whether it is possible to have an upper bound of type:

$$\mu_1^{[p]}(M) \leq c(n,p)$$

for a constant depending only on p and n, that is: is $\mu_1^{[p]}$ bounded above uniformly in the class of spherical minimal immersions? Note that the above corollary would give such a bound if the functional

$$\frac{1}{\mathrm{vol}(M)} \int_M scal$$

is bounded below uniformly (in the class of spherical minimal immersions); however this is not true, at least in dimension 2, due to examples given by Lawson of minimal immersions in $\mathbb{S}^3$ having arbitrarily large genus and volume bounded above.

Minimal immersions into a sphere are of greatest interest. A result of Takahashi states that, if M^n is minimal into $\mathbb{S}^{m+n}$, then any coordinate function restricts to an eigenfunction on M^n associated to the eigenvalue n. Therefore $\mu_1^{[0]}(M^n) \leq n$, and the well-known Yau conjecture asserts that in fact, if M^n is a minimal *embedded hypersurface*, then $\mu_1^{[0]}(M^n) = n$.

Choi and Wang [5] proved in that case the inequality:

$$\mu_1^{[0]}(M^n) \geq \frac{n}{2}.$$

It would be interesting to see whether a lower bound of type:

$$\mu_1^{[p]}(M^n) \geq c'(n,p) > 0$$

is possible in a suitable class of spherical minimal embeddings.

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