FINSLER SPACES CORRESPONDING TO DISTANCE SPACES *

LAJOS TAMÁSSY

Department of Mathematics, University of Debrecen, P. O. Box 12, H-4010 Debrecen, Hungary

E-mail: tamassy@math.klte.hu

In a connected Finsler space $F^n = (M, \mathcal{F})$ any point $q \in M$ has a distance $\varrho^F(p, q)$ from another point $p \in M$. Thus a Finsler space $(M\mathcal{F})$ determines a distance space $(M, \varrho^F) : \mathcal{F} \mapsto \varrho^F$. By a theorem of Busemann and Mayer, from this ϱ^F one can reconstruct the Finsler metric $\mathcal{F}$, and thus the Finsler space $(M, \mathcal{F}) : \varrho^F \mapsto \mathcal{F}$. But not every distance space (M, ϱ) does induce a Finsler space $(M, \mathcal{F})$, and even if it does: $\varrho \mapsto \mathcal{F}$, it is not sure that the ϱ^F induced by the $\mathcal{F}$ obtained equals the initial $\varrho : \varrho \mapsto \mathcal{F} \mapsto \varrho^F \overset{?}{=} \varrho$.

We investigate the relations between Finsler and distance spaces. We look for answers to questions as: When and how does a distance space induce a Finsler space, and what are the characteristics of these distance spaces? When does the process $\varrho \mapsto \mathcal{F} \mapsto \varrho^F$ lead to the original ϱ (when is ϱ a ϱ^F)? Longer proofs are omitted.

1. Introduction: Finsler spaces, distance spaces

1. A *Finsler space* $F^n = (M, \mathcal{F})$ is an n-dimensional manifold M equipped with a Finsler metric (structure function, fundamental function) $\mathcal{F} : TM \to R^+ = [0, \infty)$, $(p, y) \mapsto \mathcal{F}(p, y)$, $p \in M$, $y \in T_pM$, satisfying the requirements:

(F i) regularity: $\mathcal{F} \in C^0$ on TM, and $\mathcal{F} \in C^\infty$ on the slit tangent bundle $TM \setminus 0 = \{(p, y) \mid y \neq 0\}$

(F ii) positive homogeneity (of degree 1): $\mathcal{F}(p, \lambda y) = \lambda \mathcal{F}(p, y)$, $\lambda \in R^+$

(F iii) strong convexity: $\frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}(p, y) v^i v^j > 0$, $i, j = 1, \dots, n$ for all non-null vectors $v \in T_pM$.

In place of (F ii) a more restrictive requirement is the

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(F iv) absolute homogeneity (of degree 1): $\mathcal{F}(p, \lambda y) = |\lambda| \mathcal{F}(p, y)$, $\lambda \in R$.

The Finsler norm of $y \in T_pM$ is defined by $\|y\|_F := \mathcal{F}(p, y)$, and the Finsler arc length of a piecewise differentiable (this will always be supposed) curve $c : [a, b] \to M$, $t \mapsto c(t)$ is given by the integral

$$s = \int_a^b \mathcal{F}(c, \dot{c}) dt = \int_a^b \langle \dot{c}, \dot{c} \rangle_{g^F_{(\dot{c})}} \, dt,$$

where the 2-form $g^F_{(Y)}$ is defined by

$$g^F_{(Y)}(U, V) = \langle U, V \rangle_{g^F_{(Y)}} := \frac{1}{2} \frac{\partial^2 \mathcal{F}^2}{\partial s \partial t} (Y + sU + tV), \quad Y, U, V \in T_pM.$$

Its local components at $p \equiv x$ and y are

$$\left(g^F_{(Y)} |_{p,y} \right)_{ij} \equiv g^F_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j} (x, y).$$

Thus in a Finsler space

$$(a) \qquad s = \int_a^b \left(g^F_{ij}(x, \dot{x}) \dot{x}^i \dot{x}^j \right)^{1/2} dt,$$

while the arc length of $c(t) \approx x(t)$ in a Riemann space $V^n = (M, g)$ is

$$(b) \qquad s = \int_a^b \langle \dot{c}, \dot{c} \rangle_g \, dt = \int_a^b \left(g_{ij}(x) x^i \dot{x}^j \right)^{1/2} dt.$$

(a) and (b) are very similar, only in the Riemann case (b) the integrand is the square root of a quadratic form in $\dot{x}$, while in the Finsler case (a) the expression under the square root need not be quadratic in $\dot{x}$. Since both Finsler and Riemann geometries are built on the arc length of curves, on the base of the strong similarity of (a) and (b) we can say that Finsler geometry is just Riemannian geometry without the quadratic restriction. This witty remark is due to S. S. Chern [6]. Indeed the two geometries have many basic similarities.

The requirement of $\mathcal{F} \in C^\infty$ in (F i) is quite natural in a differential geometry.

(F ii–iv) have simple and important geometric meaning. (F ii) is equivalent to the invariance of the arc length against orientation − preserving reparametrizations of the curve. Also in consequence of (F ii) the graph of $z = \mathcal{F}(p_0, y)$ in $R^{n+1} = (T_{p_0}M)(y) \times R(z)$ is a cone centered at p_0. The orthogonal projection in R^{n+1} on $T_{p_0}M$ of the intersection of the cone $z = \mathcal{F}(p_0, y)$ and the hyperplane $z = 1$ is $\mathcal{I}(p_0) := \{y \in T_{p_0}M \mid \mathcal{F}(p_0, y) = 1\}$.

This is the indicatrix of F^n at p_0. $\mathcal{I}(p)$ plays a role similar to that of the unit sphere of the Euclidean space E^n.

(F iii) is equivalent to the triangle inequality in T_pM with respect to the Finsler norm:

$$(F\ iii')\ \ \mathcal{F}(p,y_1)+\mathcal{F}(p,y_2)>\mathcal{F}(p,y_1+y_2),\quad \forall\, y_1,y_2\in T_pM,\quad y_2\neq \lambda y_1.$$

$T_{p_0}M$ endowed with the Finsler norm $\|y\|_F = \mathcal{F}(p_0,y)$ is a Minkowski space $\mathcal{M}^n = (T_{p_0}M, \mathcal{F}(p_0,y))$. So a Finsler space makes any of its tangent space T_pM into a Minkowski space. In the case of a Riemannian space V^n the indicatrices $\mathcal{I}$ are ellipsoids, and the induced Minkowski space are Euclidean spaces.

$\mathcal{F}\in C^0$ at $y=0$ is a consequence of (F ii) and $\mathcal{F}\in C^\infty$ on $TM\backslash 0$. However, in view of (F ii, iii) more cannot be a achieved. Indeed, if we had $\mathcal{F}\in C^1$ at $y=0$, then $z=\mathcal{F}(p_0,y)$ (which is a cone in $(T_{p_0}M)(y)\times R(z)=R^{n+1}$ in consequence of (F i, ii)) would be a hyperplane through the origin of R^{n+1}. Since $z\geq 0$ (because of $\mathcal{F}: TM\to R^+$) this hyperplane would be $T_{p_0}M$. Then $z=\mathcal{F}(p_0,y)\equiv 0$, which is not compatible with (F iii).

Replacing the positive homogeneity (F ii) by the absolute homogeneity (F iv) yields important special spaces. At the early stage of Finsler geometry, (F iv) was usually supposed. (F iv) is equivalent to the invariance of the Finsler arc length s against any reparametrization of the curves including the change of orientation. It is also equivalent to $\mathcal{F}(p,y)=F(p,-y)$, $\forall p,y$. In this case $\mathcal{M}^n=(T_{p_0}M,\mathcal{F}(p_0,y))$ is a Banach space.

2. A *distance space* (M,ϱ) is a set M and a distance function $\varrho: M\times M\to D$ associating to any ordered pair p, q an element $\varrho(p,q)$ of the "distance set" D. In most cases, as in our case too, D consists of the non-negative reals or a subset of them. If ϱ has still the properties: a) $\varrho(p,q)=0\Longleftrightarrow p=q$ (positive definiteness), b) $\varrho(p,q)=\varrho(q,p)$ (symmetry), and c) $\varrho(p,q)+\varrho(q,r)\geq\varrho(p,r)$ (triangle inequality), then (M,ϱ) is called a metric space ([1] Section 8). If c) may fail, then ϱ and also (M,ϱ) are semi-metric. They are genuine semi-metric if c) really fails. If a) and c) are satisfied, but b) may fail, then ϱ and (M,ϱ) are called quasi-metric [9] (or genuire quasi-metric if b) really fails).

Distance spaces were introduced by K. Menger, and developed by L. Blumenthal, H. Busemann, M. Fréchet and others. Distance spaces were used in investigations of geometric problems without differentiability conditions [2]. They often appear also in recent topological studies, e.g. in investigations on the metrizability of topological spaces, etc. ([7], [8], [9], [11]).

2. Induced distance functions: $\mathcal{F} \mapsto \varrho^F$

Let $\Gamma_{(p,q)}$, $p, q \in M$ be the collection of all equally oriented cures $c(t)$, $a \leq t \leq b$ of a connected manifold M emanating from p and terminating at q. Then a Finsler space $F^n = (M, \mathcal{F})$ determines by

$$(1) \qquad \varrho^F(p,q) = \inf \int_{\Gamma(p,q)} \mathcal{F}(c, \dot{c})dt, \quad c(a) = p, \quad c(b) = q$$

a distance function ϱ^F. (1) is a correspondence

$$\mathcal{F}(x,y) \mapsto \varrho^F(p,q),$$

which orders a distance space to a Finsler space:

$$F^n = (M, \mathcal{F}) \mapsto (M, \varrho^F).$$

We want to collect some properties of this ϱ^F (cf. [4] Chap. 6, eps. sec. 6.4). Clearly

$$(R\ i) \qquad \varrho^F(p,q) \geq 0 \quad \text{and} \quad \varrho^F(p,q) = 0 \iff p = q$$

(the positive definiteness of ϱ^F). Nevertheless, without the absolute homogeneity (F iv), $\varrho^F(p,q)$ may differ from $\varrho^F(q,p)$. To the symmetry:

$$(R\ ii) \qquad \varrho^F(p.q) = \varrho^F(q,p)$$

the absolute homogeneity (F iv) is necessary and sufficient.
Furthermore ϱ^F satisfies the triangle inequality

$$(R\ iii) \qquad \varrho^F(p,q) + \varrho^F(q,r) \geq \varrho^F(p,r), \quad p,q,r \in M.$$

By (1) there exist curves $c_1(t)$, $0 \leq t \leq 1$ from p to q, and $c_2(t)$, $1 \leq t \leq 2$ from q to r, such that

$$\dot{I}_1 = \int_0^1 \mathcal{F}(c_1, \dot{c}_1)dt = \varrho^F(p,q) + \varepsilon \ \text{ and } \ \dot{I}_2 = \int_1^2 \mathcal{F}(c_2, \dot{c}_2)dt = \varrho^F(q,r) + \varepsilon$$

for arbitrary small $0 < \varepsilon$. Then the arc length $\dot{I}_3$ of $c_3(t) = c_1 \cup c_2$, $0 \leq t \leq 2$ is

$$\dot{I}_3 = \int_0^2 \mathcal{F}(c_3, \dot{c}_3)dt = \varrho^F(p,q) + \varrho^F(q,r) + 2\,\varepsilon.$$

Since $c_3 \in \Gamma_{(p,r)}$, we obtain

$$\varrho^F(p,r) = \inf \int_{\Gamma(p,r)} \mathcal{F}(c, \dot{c})dt \leq \dot{I}_3.$$

Thus $\varrho^F(p,r) \le \varrho^F(p,q) + \varrho^F(q,r) + 2\varepsilon$, $\forall\, \varepsilon > 0$. This yields (R iii). (R iii) does not need the absolute homogeneity of $\mathcal{F}$.

The triangle inequality (R iii) does not need (F iii). Suppose that $\mathcal{F}$ does not satisfy (F iii) or (F iii'). (Then $(M, \mathcal{F})$ is not a Finsler space in our sense.) This happens if the indicatrices $\mathcal{F}(p_0, y) = 1$, $p_0 \in M$ are star-shaped, smooth, but non-convex. Arc lengths s of curves and distance functions ϱ^F can be formed even in this case, and our considerations described in the previous paragraph also remain alive. Thus (R iii) too is valid. It means also that neither of these distance functions can be genuine semi-metric. – Nevertheless we can present differential geometric examples for proper semi-metric spaces, if ϱ is given in another way. Let us consider a Minkowski space $\mathcal{M}^n = (R^n, \mathcal{F})$ in an adapted coordinate system (x) with a symmetric, star-shaped, smooth and non-convex indicatrix $\mathcal{I}$, and define a distance function $\varrho(x_1, x_2)$ by the Minkowski norm of the vector $\overrightarrow{x_1, x_2}$:

$$\varrho(x_1, x_2) := \|\overrightarrow{x_1 x_2}\|_M.$$

Then (R ii) is satisfied because of the symmetry of $\mathcal{I}$, but the triangle inequality (R iii) is not, since $\mathcal{I}$ is non-convex.

So (M, ϱ^F) is a metric space provided $\mathcal{F}$ is absolutely homogeneous, and it is a genuine quasi-metric space if $\mathcal{F}$ is only positively homogeneous. Further on (M, ϱ) is supposed to be quasi-metric. Metric (M, ϱ) are included as special case.

Using in F^n a geodesic polar coordinate system (r, φ) in a neighbourhood $U \subset M$ around p_0, we find that $\varrho^F(p_0, q) = r$. This shows that $\varrho^F(p_0, q) \in C^0$ at $q = p_0$, $\varrho^F(p_0, q) \notin C^1$ at $q = p_0$, and $\varrho^F(p_0, q) \in C^\infty$ on the punctured $U \setminus 0$ $(r \neq 0)$.

Let $q(t)$, $0 \le t \le a$ be a geodesic of F^n with $q(0) = p_0$ and $\lim\limits_{t\to 0} \dot{q}(t) = y_0 \neq 0$. Then

$$(2') \quad \lim_{t\to 0}\left[\frac{d}{dt}\varrho^F(p_0, q(t))\right] = \lim_{t\to 0}\left[\frac{d}{dt}\int_0^t F(q(\tau), \dot{q}(\tau))d\tau\right] = \mathcal{F}(p_0, y_0) > 0.$$

$\frac{d}{dt}\varrho^F(p_0, q(t)) = \frac{d}{dt}\big|_{q(t),\dot{q}(t)}\varrho^F(p_0, q)$ is the directional derivative of ϱ^F at $q(t)$ in the direction $\dot{q}(t)$. Since directional derivatives depend on the point and the direction only, $q(t)$ in (2') can be replaced by any other $c(t)$, $0 \le t$ emanating from $p_0 = c(0)$, and having at p_0 the (one sided) tangent y_0. Then

$$(2) \qquad \lim_{t\to 0}\left[\frac{d}{dt}\varrho^F(p_0, c(t))\right] = \mathcal{F}(p_0, y_0), \qquad y_0 = \lim_{t\to 0}\frac{dc}{db}.$$

(2) is basically the content of the Busemann-Mayer theorem ([5] p. 186, in a more comfortable form in [4] p. 153, or [10] p. 72).

Thus we obtain:

(R iv) (a) $\varrho^F(p_0, q) \in C^0$ at $q = p_0$

(b) $\varrho^F(p, q) \in C^\infty$ in an open domain around, but without p_0.

(c) There exists $\lim\limits_{t\to 0} \frac{d}{dt}\big|_{c(t),\dot c(t)} \varrho^F(p_0, q)$ for any $(c(t),\ 0 \le t \le a$ emanating from $p_0 = c(0)$. The value of this limit is $\mathcal{F}(p_0, y_0)$, $y_0 = \lim\limits_{t\to 0} \frac{dc}{dt}$, which is positive if $y_0 \neq 0$, of class C^∞ at $p_0, y_0 \neq 0$, and of class C^0 if $y_0 = 0$.

It follows from the properties of the directional derivatives that

$$(R\ v) \qquad \lim_{t\to 0} \frac{d}{dt}\Big|_{\bar c(t),\dot{\bar c}(t)} \varrho^F(p_0, q) = \lambda \lim_{t\to 0} \frac{d}{dt}\Big|_{c(t),\dot c(t)} \varrho^F(p_0, q), \qquad \lambda \in R^+$$

if $\bar c(0) = c(0)$ and $\dot{\bar c}(t) = \lambda \dot c(0)$. (R v) is a consequence of (R iv).

Let $c_1(t),\ c_2(t),\ c_3(t),\ 0 \le t$ be curves emanating from p_0 with non-null and non-parallel tangents $\dot c_1(0) = y_1,\ \dot c_2(0) = y_2,\ \dot c_3(0) = y_1 + y_2$. By (2), (F iii) and (R iv, c) we obtain

$$(R\ vi) \qquad \begin{aligned} &\lim_{t\to 0} \frac{d}{dt}\Big|_{c_1(t),\dot c_1(t)} \varrho^F(p_0, q) + \lim_{to\to 0} \frac{d}{dt}\Big|_{c_2(t),\dot c_2(t)} \varrho^F(p_0, q) > \\ &\qquad > \lim_{t\to 0} \frac{d}{dt}\Big|_{c_3(t),\dot c_3(t)} \varrho^F(p_0, q). \end{aligned}$$

This is somewhat stronger than the local triangle axiom (see [13] p. 56)

(R iv–vi) hold also for $\varrho^F(q, p_0)$.

We can summarize these statements in

Proposition 2.1 *The distance function ϱ^F derived from an F^n by (1) possesses the properties (R i, iii–vi). (R ii) is added iff $\mathcal{F}$ is absolutely homogeneous.*

3. Induced Finsler spaces: $\varrho \mapsto \mathcal{F}$

Further on we suppose that ϱ in place of ϱ^F satisfies (R i, iii-vi).

We want to define a correspondence

$$(3) \qquad \varrho(p_0, q) \mapsto \bar{\mathcal{F}}(p_0, y), \quad \forall\, p_0 \in M; \quad (M, \varrho) \mapsto (M, \bar{\mathcal{F}})$$

with the natural requirement that in case of $\varrho = \varrho^F$ the Finsler metric $\bar{\mathcal{F}}$ corresponding to $\varrho = \varrho^F$ by (3) is just that $\mathcal{F}$ from which ϱ^F originates by (1):

$$\left(\mathcal{F} \overset{(1)}{\longmapsto}\right) \varrho^F \overset{(3)}{\longmapsto} \bar{\mathcal{F}} = \mathcal{F}.$$

We know that between ϱ^F and F the relation (2) subsists. Hence (3) must have the form

$$\bar{\mathcal{F}}(p, y) := \lim_{t \to 0} \left[\frac{d}{dt} \varrho(p, c(t)) \right], \qquad y = \lim_{t \to 0} \frac{dc}{dt}, \tag{4}$$

where $c(t)$, $0 \leq t \leq a$, $c(0) = p$ is a curve emanating from p. (4) is meaningful, since the limit exists by our assumption (R iv,c).

$\bar{\mathcal{F}}$ defined by (4) is a Finsler metric. By (R iv,c) $\bar{\mathcal{F}}(p, y)$ is non negative, it is of class C^∞ if $\dot{c}(0) = y \neq 0$, and of class C^0 if $\dot{c}(0) = y = 0$. Thus $\bar{\mathcal{F}}(p, y)$ of (4) satisfies (F i). By (R v) it satisfies (F ii). Finally by (2) and (R vi) it satisfies also (F iii). Thus we obtain

Proposition 3.1 *If $\varrho(p, q)$ satisfies (R i, iii–vi), then $\bar{\mathcal{F}}(p, y)$ defined by (4) is a Finsler metric. If also (R ii) is satisfied, then $\bar{\mathcal{F}}$ is absolutely homogeneous.*

Without any of the conditions (R i, iii–vi) on ϱ $\bar{\mathcal{F}}(p, y)$ defined by (4) may not be a Finsler metric.

By (1) and (4) $\mathcal{F} \overset{(1)}{\longmapsto} \varrho^F \overset{(4)}{\longmapsto} \mathcal{F} \overset{(1)}{\longmapsto} \varrho^F$. This means that (1) and (4) are map and inverse map. Thus they induce between $\{\mathcal{F}\}$ and $\{\varrho^F\}$ (over a given M) a $1:1$ relation. Nevertheless (4) orders to every ϱ (which satisfies (R i, iii–vi) an $\mathcal{F}$, and thus $\varrho \overset{(4)}{\longmapsto} \mathcal{F} \overset{(1)}{\longmapsto} \varrho^F$. We show that in this sequence $\varrho^F \neq \varrho$ may occur. This fact is expressed by the

Theorem 3.1 *$\{\varrho^F\}$ is a proper part of $\{\varrho\}$, where ϱ satisfies (R i, iii–vi).*

This can be proved by giving an example, where ϱ induces by (4) a Finsler metric $\mathcal{F}(p, y)$, yet the ϱ^F obtained from this $\mathcal{F}$ by (1) differs from the initial $\varrho : \varrho^F \neq \varrho$.

First we give a 1-dimensional example. Let $M = R^1 = R$ be the Euclidean line E^1, and (x) the canonical coordinates on it. Let $\varrho(0, x)$, $x \in [0, \infty)$ be a strictly increasing C^∞ function with strictly decreasing first derivative with $\varrho(0, 0) = 0$, and satisfying

$$\lim_{x \to 0^+} \frac{d}{dx} \varrho(0, x) = 1 \tag{5}$$

(e.g. $\varrho(0,x) = \ln(x+1)$). We define ϱ for $\bar{x} < 0$ by

$$\varrho(0,\bar{x}) = \varrho(0,|\bar{x}|), \tag{6}$$

and for $x_0 \neq 0$ by

$$\varrho(x_0,x) = \varrho(0,x-x_0). \tag{7}$$

The functions $\varrho(x_0,x)$ for different x_0 are parallel translates of each other. One can prove that they satisfy (R i–vi). In consequence of (R i–iii) (M,ϱ) is a metric space. According to Proposition 3.1 this ϱ generates by (4) a Finsler space $F^1 = (R^1, \mathcal{F})$, ((F iii) with the sign of equality). By (4), (6) and (7) $\mathcal{F}(x_0,a) = \mathcal{F}(x_0,-a)$. Thus $\mathcal{F}$ is absolutely homogeneous. By (4) and (7) $\mathcal{F}(x,a)$ is independent of x. Therefore F^1 is a Minkowski space with symmetric indicatrix, and because of $n=1$ it is Euclidean space E^1. hence $\varrho^F(x_1,x_2) = |x_1 - x_2|$. Nevertheless, by the integral mean theorem

$$\varrho(x_1,x_2) = \int_{x_1}^{x_2} \varrho'(x_1,z)dz = |x_1-x_2|\varrho'(x_1,z^*), \quad z^* \in (x_1,x_2).$$

By (5), (7) and the strict decrease of $\varrho'(x_1,z)$ on $z > x_1$ we obtain

$$\lim_{z \to x_1^+} \varrho'(x_1,z) = 1 > \varrho'(x_1,z^*).$$

Thus

$$\varrho(x_1,x_2) < |x_1-x_2| = \varrho^F(x_1,x^2), \quad \text{i.e. } \varrho \overset{(4)}{\longmapsto} \mathcal{F} \overset{(1)}{\longmapsto} \varrho^F \neq \varrho.$$

The example discussed can be extended to $M{=}R^n(x)$. Let us define $z{=}\varrho(0,x)$ over each ray $x^i = r^i t$, $0 \leq t$, $\Sigma(r^i)^2 = 1$, emanating from the origin 0, as in the previous paragraph. Then $z = \varrho(0,x)$ is a surfaces of revolution Θ around the z axis in $R^{n+1}(x,z)$. We define $\varrho(x_0,x)$, $x_0 \neq 0$ by (7). We can prove again that the initial distance ϱ differs from ϱ^F derived from the F^n obtained from ϱ by (4): $\varrho \overset{(4)}{\longmapsto} \mathcal{F} \overset{(1)}{\longmapsto} \varrho^F \neq \varrho$. The line of reasoning is similar to the previous case, but the proofs are somewhat more involved.

Similar examples can be constructed on manifolds M different from R^n, provided that M admits a locally Minkowski structure. This is possible iff M admits an open cover $M = \bigcup_\alpha U_\alpha$ by local carts, and on each U_α there exists a coordinate system (x_α), such that the transitions $(x_\alpha) \longleftrightarrow (x_\beta)$ on $U_\alpha \cap U_\beta$ are linear ([12] Section 2). The torus has this property, but the sphere does not ([3] p. 250; [4] p. 14).

4. Conditions for $\varrho = \varrho^F$

Further on we suppose that in $F^n = (M, \mathcal{F})$ any pair of points $p, q \in M$ is connected by a (short) geodesic $g(t)$, $a \le t \le b$, $g(a) = p$, $g(b) = q$ whose arc length is $\varrho^F(p, q)$. This is certainly true if F^n is geodesically complete. (In this case the infimum in (1) is a minimum.)

Starting with an arbitrary ϱ (which satisfies (R i, iii–vi)), it may happen that

$$\varrho \overset{(4)}{\longmapsto} \mathcal{F} \overset{(1)}{\longmapsto} \varrho^F \neq \varrho$$

(i.e. ϱ^F may differ from ϱ), as it was shown by the example of the previous section. We look for conditions assuring

$$\varrho \overset{(4)}{\longmapsto} \mathcal{F} \overset{(1)}{\longmapsto} \varrho^F = \varrho.$$

First we show the parallelity of certain vector fields. Let $g(t)$, $t \in [0, T]$ be a short geodesic of $F^n = (M, \mathcal{F})$ from p_0 to q. Such a geodesic exists, for F^n is complete. Then for any t_1, $0 < t_1 < t < T$

$$\varrho^F(p_0, g(t)) = \varrho^F(p_0, g_1) + \varrho^F(g_1, g(t)), \quad g_1 = g(t_1).$$

From this

$$\left[\frac{d}{dt}\varrho^F(p_0, g(t)\right]_{|t_1} = \lim_{t \to t_1^+}\left[\frac{d}{dt}\varrho^F(g_1, g(t))\right]. \tag{8}$$

Consider the distance surface of F^n attached to p_0

$$\Theta^F_{p_0} : z = \varrho^F(p_0, q) \subset U(q) \times R^1(z),$$

where $U(q) \subset M$ is a coordinate neighbourhood of p_0. p_0 is the vertex (cape) of $\Theta^F_{p_0}$. The curve $\xi_0(t) := (g(t), \varrho^F(p_0, g(t)) \subset U \times R^+$ lies on $\Theta^F_{p_0}$, and $\xi_1(t) := (g(t), \varrho^F(g_1, g(t)) \subset \Theta^F_{g_1}$. By (8) their tangents: $\dot{\xi}_0(t_1)$ and $\lim_{t \to t_1^+} \dot{\xi}_1(t) =: \dot{\xi}_1^+(t_1)$ are parallel:

Proposition 4.1 $\dot{\xi}_0(t_1) \| \dot{\xi}_1^+(t_1), \ \forall\, t_1 \in (0, T).$

Consider the projection $\pi : U \times R^+ \to U$, $(p, z) \mapsto p$. Then

$$d\pi\dot{\xi}_0(t_1) = d\pi\dot{\xi}_1^+(t_1) = \dot{g}(t_1),$$

and $\dot{\xi}_0(t_1)$ and $\dot{\xi}_1^+(t_1)$ are the lifts of $\dot{g}(t_1)$ to $T_{\xi_0(t_1)}\Theta^F_{p_0}$ resp. $\lim_{t \to t_1^+} T_{\xi_1(t)}\Theta^F_{g_1}$.

In a distance space with (R i, iii–vi) the notion of geodesic can be replaced to a certain extent by that of "parallelity curve". Let us consider a curve

$p(t)$, $t \in [0,T]$. Along this there exist distance surfaces $\theta^{\varrho}_{p(t)} : z = \varrho(p(t), q)$ and curves $\zeta_0(t)$, $\zeta_1(t)$ on them over $p(t)$ similarly to $\xi_0(t)$ and $\xi_1(t)$. If $\zeta_0(t) \in C^1$, and

$$\dot{\zeta}_0(t_1) \| \dot{\zeta}_1^+(t_1), \quad \forall\, t_1 \in (0,T),$$

then $p(t)$ is called a *parallelity curve.*

Theorem 4.1 *For any curve $c(t)$, $t \in [0,T]$ of a distance space (M,ϱ) and for the Finsler metric $\mathcal{F}$ determined by ϱ according to (4) we obtain*

(a) $\varrho(c(0), c(T)) \le \int_0^T \mathcal{F}(c,\dot{c})dt$

(b) if $c(t)$ is a parallelity curve, then

$$\varrho(c(\tau), c(t)) = \int_\tau^t \mathcal{F}(c,\dot{c})du, \quad 0 \le \tau < t < T \tag{9}$$

(c) if along $c(t)$ (9) holds for $\forall \tau, t$, $0 \le \tau < t < T$, then $c(t)$ is a parallelity curve.

Proofs of Theorems 4.1, 4.2 and 4.3 are omitted.

Corollary 4.1 *In a Finsler space parallelity curves and short geodesics coincide.*

As we have shown, a distance space (M,ϱ) with (R i, iii-vi) determines an $F^n = (M,\mathcal{F})$, and this F^n determines a ϱ^F:

$$\varrho \overset{(4)}{\longmapsto} \mathcal{F} \overset{(1)}{\longmapsto} \varrho^F. \tag{10}$$

Theorem 4.2 *In (10) $\varrho^F = \varrho$ iff any short geodesic of F^n (determined by ϱ) is a parallelity curve of (M,ϱ).*

In other words: the distances in (M,ϱ) coincide with the distances of a Finsler space iff the short geodesics of the Finsler space are parallelity curves of the distance space.

In Theorem 4.2 we required the parallelity property on curves determined by F^n, and not by the distance space (M,ϱ). No we replace the parallelity property (the condition of Theorem 4.2) by another, which is expressed directly in terms of the (M,ϱ).

Let us consider two points a, b of a distance space (M,ϱ), and a sphere $S^{\varrho}_a(t) := \{q \mid \varrho(a,q) = t\}$ around a with radius $t \le r = \varrho(a,b)$. Then there exists another sphere $S^{\varrho}_b(\tau) := \{q \mid \varrho(q,b) = \tau\}$ such that the two spheres osculate each other from outside at a point $\sigma(t) \in S_a^\varrho(t)$ and $S_b^\varrho(\tau)$. If $\sigma(t)$, $t \in [0, r]$ is a C^1 curve, then it will be called *osculation curve*, and we obtain

Theorem 4.3 *For a distance space* (M, ϱ) *(where* ϱ *satisfies (R i, iii–vi)) and for a* ϱ^F *determined by a Finsler space by (1) we obtain* $\varrho = \varrho^F$ *iff any osculation curve is a parallelity curve in* (M, ϱ).

Because of the triangle inequality (R iii) $r \le t + \tau$ for any osculation curve $\sigma(t; a, b)$. If $r = t + \tau$, $\forall\, 0 < t < r$, then $\sigma(t; a, b)$ is called straight ([3]) or a Hilbert curve ([5] p. 170). In a Finsler space osculation curves are short geodesics.

References

1. L. M. Blumenthal, *Theory and Application of Distance Geometry*, Clarendon Press, Oxford, 1953.
2. W. Benz, *Reelle Abstandräume und hyperbolische Geometrie*, Results. Math. **34** (1998), 56–68.
3. D. Bao and S. S. Chern, *A note on the Gauss-Bonnet theorem for Finsler spaces*, Ann. Math. **143** (1996), 233-252.
4. D. Bao, S. S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer, New York, 2000.
5. H. Busemann and W. Mayer, *On the foundation of calculus of variation*, Trans. AMS. **49** (1948), 173–198.
6. S. S. Chern, *Finsler geometry is just Riemanian geometry without the quadratic restriction*, Notices Amer. Math. Soc. **43** (1996), 959–963.
7. H. P. A. Künzi, *Nonsymmetric distances and their associated topologies: About the origins of basic ideas*, In: Handbook of the History of General Topology, ed by C. E. Aull and L. Lowen, vol. 3, Kluwer, Dordrecht, (2001), 853–968.
8. T. G. Raghavan and I. L. Reilly, *Metrizability of quasi-metric spaces*, J. London Math. Soc. **15** (1977), 169–172.
9. S. Romaguera and M. Schellekens, *Quasi-metric properties of complexity spaces*, Topolgy Appl. **98** (1999), 311–322.
10. Z. Shen, *Lecture notes on Finsler geometry*, Indiana Univ. 1998.
11. R. A. Stoltenberg, *On quasi-metric spaces*, Duke Math. J. **36** (1969), 65–71.
12. L. Tamássy, *Point Finsler spaces with metrical linear connections*, Publ. Math. Debrecen **56** (2000), 643-655.
13. P. Waszkiewicz, *The local triangle axiom in toplogy and domain theory*, Appl. Gen. Top. **4** (2003), 47–70.