ON SURFACES OF ROTATION OF A GIVEN CONSTANT GAUSSIAN CURVATURE AND THEIR VISUALIZATIONS *

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This paper deals with the surfaces of rotation with a given constant Gaussian curvature and their representation by our software package. We give a survey of the theory of these surfaces. Their visualizations involve an extension of Malkowsky's software package [12] for the visualization of mathematics [4, 5, 9, 10, 11, 13]. We also developed a Windows application which works independently of any other software package, and describe its user interface.

1. Introduction

We developed a software package [12, 10, 11, 13] in Borland PASCAL and DELPHI for visualizations and animations, mainly in the field of classical differential geometry.

The main stream to represent surfaces in modern computer graphics is their approximation by the faces of polyhedra, mainly by triangulation or rectangulation. This approach yields very fast results, allows certain transformations such as translations, rotations or scalings in real time. It also enables fast transmissions on the Internet.

One problem of this approach is that the approximating polyhedra are not smooth. It leads to the appearance of undesirable edges which do not exist in reality, and shows clearly when the surfaces are printed. This disadvantage can be overcome by increasing the number of the polyhedra or by rendering, for instance by shading or applying an illumination model. These techniques give very good results. The time needed, however, increases to

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a multiple due to the much larger number of computations involved, in particular for pictures of large sizes. Consequently those techniques are normally applied at the end of the work when printing quality or smooth appearance is needed.

Another problem in the approximation of surfaces by polyhedra arises in the representation of curves on surfaces; they appear as if they are not really on the surface. We are mainly interested in this task, since it is important in differential geometry to be able to visualise certain classes of curves on surfaces, such as asymptotic or geodesic lines, lines of curvature, level lines, or the lines of intersection of surfaces.

This is why we use line graphics, and represent surfaces by families of lines on them. We do not apply any rendering. Invisible parts of curves can either be dotted or not be drawn at all. Our drawing process is slower than that for the approximation by polyhedra, because we use many points for each line. But the result has printing quality. It is, however, much faster than rendering, because we do not need to compute the values of every pixel or dot of the body.

This approach is first introduced by Endl [2, 3] in the late eighties of the last century for some elementary geometrical bodies such as Platonic bodies, cones, cylinders or spheres. From the early nineties, Malkowsky used the same approach for many topics in differential geometry [12]. Later several extensions were made in different fields, for instance in functional analysis, topology, physics, chemistry and the engineering sciences [4, 5, 9, 10, 11, 13].

This paper involves an extension of Malkowsky's software to the visualization of surfaces of rotation with a given constant Gaussian curvature. The theory of these surfaces is outlined in Sections 2 and 3. It can be found in many text books on differential geometry [1, 6, 7, 8]. We also developed a Windows application which is independent of any other software. Its user interface is described in Section 4. All the geometrical figures in this paper have been created by our software package.

2. Background and Notations

First we recall a few basic notations and results. Let S be a surface in $\mathbb{R}^3$ given by a parametric representation

$$\vec{x}(u^i) = (x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2)) \ ((u^1, u^2) \in D)$$

where $D \subset \mathbb{R}^2$ is a domain and the component functions $x^k : D \to \mathbb{R}$ ($k = 1, 2, 3$) are of class $C^r(D)$, that is they have continuous partial derivatives

of order $r \geq 2$ on D. We use the standard notations

$$\vec{x}_k = \frac{\partial \vec{x}}{\partial u^k} \quad \text{and} \quad \vec{x}_{kj} = \frac{\partial^2 \vec{x}}{\partial u^k \partial u^j} \text{ (k,j=1,2)},$$

and assume that the vectors $\vec{x}_1$ and $\vec{x}_2$ are linearly independent on D. Then the surface normal vectors of S

$$\vec{N}(u^i) = \frac{\vec{x}_1(u^i) \times \vec{x}_2(u^i)}{\|\vec{x}_1(u^i) \times \vec{x}_2(u^i)\|}$$

exist for all $(u^1, u^2) \in D$. The functions $g_{jk}, L_{jk} : D \to \mathbb{R}$ with

$$g_{jk}(u^i) = \vec{x}_j(u^i) \bullet \vec{x}_k(u^i) \quad \text{and} \quad L_{jk}(u^i) = \vec{x}_{jk}(u^i)\vec{N}(u^i) \ (j, k = 1, 2)$$

are called the first and second fundamental coefficients of S; we write $g = g_{11}g_{22} - g_{12}^2$ and $L = L_{11}L_{22} - L_{12}^2$. Finally, the function $K : D \to \mathbb{R}$ with

$$K(u^i) = \frac{L(u^i)}{g(u^i)} \ ((u^1, u^2) \in D)$$

is called the Gaussian curvature of S.

Now let γ be a curve with a parametric representation $\vec{x}(s) = (r(s), 0, h(s))$ where $s \in I \subset \mathbb{R}$ is the arc length along γ, and $r(s) > 0$ and $|r'(s)| + |h'(s)| > 0$ on I. Furthermore, let $RS(\gamma)$ be the surface of rotation generated by rotating γ around the x^3–axis. Putting $u^1 = s$ and denoting the angle of rotation by u^2, we obtain the following parametric representation for $RS(\gamma)$ on $D = I \times (0, 2\pi)$

$$\vec{x}(u^i) = (r(u^1)\cos u^2, r(u^1)\sin u^2, h(u^1)). \tag{2.1}$$

The first and second fundamental coefficients of $RS(\gamma)$ depend on the parameter u^1 only, and are given by

$g_{11}(u^1) = (r'(u^1))^2 + (h'(u^2))^2 = 1$ (since u^1 is the arc length along γ),
$g_{12}(u^1) = 0, \ g_{22}(u^1) = r^2(u^1),$
$L_{11}(u^1) = r'(u^1)h''(u^1) - r''(u^1)h'(u^1), \ L_{12}(u^1) = 0,$
$L_{22}(u^1) = r(u^1)h'(u^1).$

Omitting the argument u^1, we obtain the Gaussian curvature of $RS(\gamma)$

$$K = \frac{(r'h'' - r''h')h'}{r},$$

and since $(r')^2 + (h')^2 = 1$ implies $r'r'' + h'h'' = 0$, this yields

$$K = \frac{r'h''h' - r''(h')^2}{r} = -\frac{((r')^2 + (h')^2)r''}{r} = -\frac{r''}{r},$$

If we assume that $K : I \to \mathbb{R}$ is a given function and write $u = u^1$, for short, then the surface of rotation that has K as its Gaussian curvature is given by

$$r''(u) + K(u)r(u) = 0 \quad \text{and} \quad h(u) = \pm \int \sqrt{1 - (r'(u))^2} \, du. \qquad (2.2)$$

We may choose the sign "+" for h without loss of generality.

The surface of rotation on the left hand side of Figure 1 is given by

$$r(t) = \int_{0.1}^{t} f(t) \, dt, \qquad f(x) = \cos\left(\frac{t^{\alpha+1}}{\alpha+1}\right)$$
$$h(t) = \int_{0.1}^{t} g(t) \, dt, \qquad g(x) = \sin\left(\frac{t^{\alpha+1}}{\alpha+1}\right)$$

where $\alpha = -0.25$, on the interval $I = (0.1, 10) \times (\pi/4, \pi)$.

The right hand side of Figure 1 illustrates three intersecting surfaces of rotation obtained from klothoids rotated around the x–, y– and z–axes. A klothoid is a planar curve with its curvature proportional to its arc length s. Its parametric representation is

$$\vec{x}(s) = \left(\int_0^s \cos \frac{cx^2}{2} \, dx, \int_0^s \sin \frac{cx^2}{2} \, dx \right). \qquad (2.3)$$

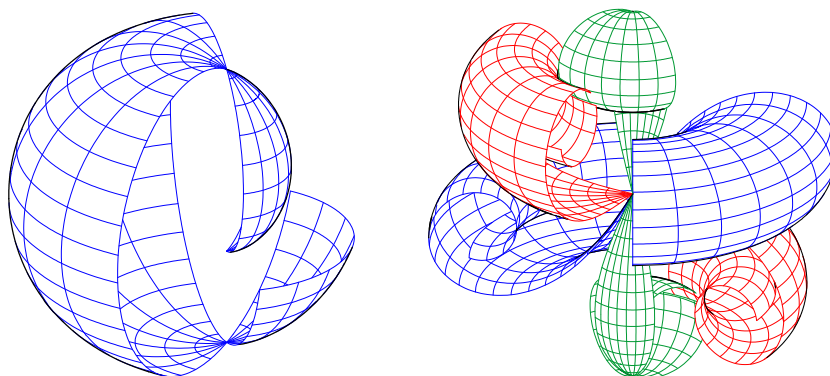Here we choose $c = \pi/2$ and $s \in I = (-5.5, 5.5)$.

Figure 1.   Surfaces of rotation

3. Spherical and Pseudo–spherical Surfaces

In this section, we discuss surfaces of rotation with a given constant Gaussian curvature; they are called *spherical* or *pseudo–spherical surfaces* depending on whether $K > 0$ or $K < 0$.

First, we assume $K = 0$. Then, it immediately follows from (2.2) that $r(u) = c_1 u + c_2$ with constants c_1 and c_2. If we choose $c_1 = 0$, then $h'(u) = \pm 1$ implies $h(u) = \pm u + d$ with some constant d, and we obtain a circular cylinder. If we choose $c_1 \neq 0$, then $(r'(u))^2 + (h'(u))^2 = 1$ implies $|c_1| \leq 1$. For $|c_1| = 1$, we have $h' \equiv 0$, hence $h \equiv const$, and we obtain a plane. For $0 < |c_1| < 1$ and a suitable choice of the coordinate system, we have $r(u) = c_1 u$ and $h(u) = d_1 u$ for some constant d_1 with $c_1^2 + d_1^2 = 1$, and we obtain a circular cone.

3.1. *Spherical surfaces*

Now we assume $K > 0$ and put $K = 1/c^2$ for some constant $c > 0$. Then the general solution of the differential equation in (2.2) is $r(u) = \lambda \cdot \cos(u/c + u_0)$ with constants λ and u_0, and, by a suitable choice of the arc length, we may assume

$$r(u) = \lambda \cos\left(\frac{u}{c}\right) \text{ with } \lambda > 0 \text{ and obtain } h(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)}\, du.$$

Thus the spherical surfaces are given by parametric representations (2.1) with

$$r(u) = \lambda\cos\left(\frac{u}{c}\right) \quad \text{and} \quad h(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)}$$
$$\text{where } \lambda > 0, \ c = \frac{1}{\sqrt{K}} \text{ for all } u \text{ with } \left|\sin\left(\frac{u}{c}\right)\right| \leq \frac{c}{\lambda}; \tag{3.1}$$

the integral for h is called *elliptic integral*. We obtain three different types of spherical surfaces corresponding to the cases $\lambda = c$, $\lambda > c$ or $\lambda < c$.

Case 1. $\lambda = \mathbf{c}$

In this case the surface has a parametric representation

$$\vec{x}(u^i) = \left(c\cos\left(\frac{u^1}{c}\right)\cos u^2, c\cos\left(\frac{u^1}{c}\right)\sin u^2, c\sin\left(\frac{u^1}{c}\right)\right)$$
$$\text{for } ((u^1, u^2) \in (-\pi/2, \pi/2) \times (0, 2\pi)).$$

This is a sphere with radius c and centre in the origin.

Case 2. $\boldsymbol{\lambda > c}$

The corresponding surfaces are called *hyperbolic spherical surfaces*. Now the integral for h in (3.1) only exists for values of u with

$$\left|\sin\left(\tfrac{u}{c}\right)\right| \le \frac{c}{\lambda}, \text{ that is } u \in I_k = \left[-c\arcsin\left(\tfrac{c}{\lambda}\right) + k\pi, c\arcsin\left(\tfrac{c}{\lambda}\right) + k\pi\right]$$
$$\text{for } k = 0, \pm 1, \pm 2, \dots \quad \text{(left in Figure 2)}.$$

Every interval I_k defines a region of the surface. The radii of the circles of the u^2–lines are minimal at the endpoints of the intervals I_k and equal to $r = \sqrt{\lambda^2 - c^2}$, whereas the maximum radius $R = \lambda$ is attained in the middle of each region (left in Figure 4).

Case 3. $\boldsymbol{\lambda < c}$

The corresponding surfaces are called *elliptic spherical surfaces*. Now the integral for h in (3.1) exists for all u and the radii r of the circles of the u^2–lines attain all values $r \le \lambda$ (right in Figure 2).

Figure 2 illustrates a hyperbolic and an elliptic spherical surface.
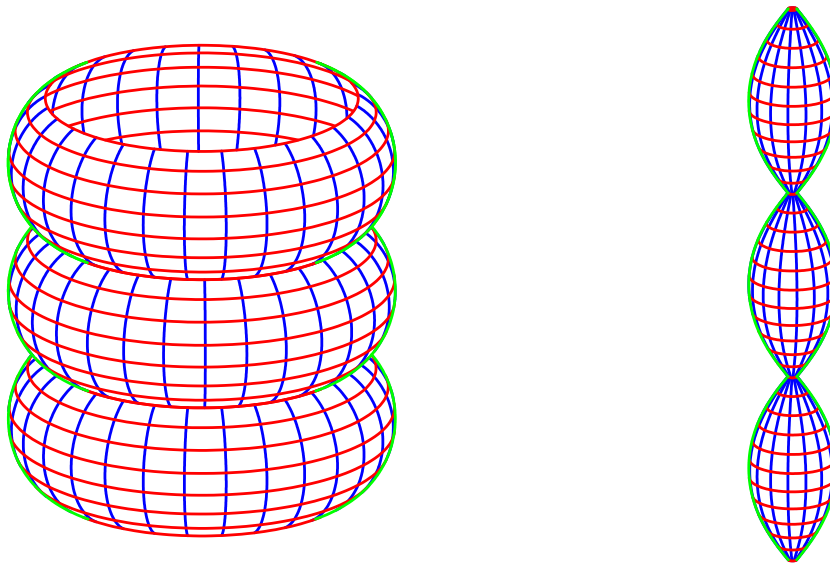

Figure 2.   Spherical surfaces

left: hyperbolic, $K = 1$, $c = 1$ and $\lambda = \dfrac{3}{2}$;   right: elliptic, $K = 1$, $c = 1$ and $\lambda = \dfrac{3}{5}$

3.2. *Pseudo–spherical surfaces*

Now we assume $K < 0$ and put $K = -1/c^2$ for some constant $c > 0$. The general solution of the differential equation in (2.2) is

$$r(u) = C_1 \cosh\left(\frac{u}{c}\right) + C_2 \sinh\left(\frac{u}{c}\right) \quad \text{with constants } C_1 \text{ and } C_2. \quad (3.2)$$

Case 1. $\mathbf{C_1} = -\mathbf{C_2} = \lambda \neq \mathbf{0}$

The corresponding surfaces are called *parabolic pseudo–spherical surfaces*. Now we obtain $r(u) = \lambda e^{-\left(\frac{u}{c}\right)}$ from (3.2), and the surface has a parametric representation (2.1) with

$$r(u^1) = \lambda \exp\left(-\frac{u^1}{c}\right) \text{ and } h(u^1) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \exp\left(-\frac{2u^1}{c}\right)}\, du^1$$
$$\text{for } u^1 > c \log\left(|\lambda|/c\right) \quad \text{(left in Figure 3).}$$

Case 2. $\mathbf{C_2} = \mathbf{0}$ and $\mathbf{C_1} = \lambda \neq \mathbf{0}$

The corresponding surfaces are called *hyperbolic pseudo–spherical surfaces*. They have a parametric representation (2.1) with

$$r(u^1) = \lambda \cosh\left(\frac{u^1}{c}\right) \quad \text{and} \quad h(u^1) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sinh^2\left(\frac{u^1}{c^2}\right)}\, du^1$$
$$\text{for } |u^1| \leq c \cdot \operatorname{arsinh}\left(\frac{c}{|\lambda|}\right) = c \log\left(\frac{c}{|\lambda|} + \sqrt{\frac{c^2}{\lambda^2} + 1}\right) \quad \text{(middle in Figure 3);}$$
$$(3.3)$$

the integral for h in (3.3) is an elliptic integral. The radii r of the circles of the u^2–lines satisfy $|\lambda| \leq r \leq \sqrt{\lambda^2 + c^2}$ (right in Figure 4).

Case 3. $\mathbf{C_1} = \mathbf{0}$ and $\mathbf{C_2} = \lambda \neq \mathbf{0}$

The corresponding surfaces are called *elliptic pseudo–spherical surfaces*. They have a parametric representation (2.1) with

$$r(u^1) = \lambda \sinh\left(\frac{u^1}{c}\right) \text{ and } h(u^1) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \cosh^2\left(\frac{u^1}{c}\right)}\, du^1$$
$$\text{for all } u^1 \text{ with } \cosh\left(\frac{u^1}{c}\right) \leq \frac{c}{|\lambda|}; \quad (3.4)$$

(since $\cosh u^1 \geq 1$ for all u^1, we must have $|\lambda| \leq c$) (right in Figure 3); the integral for h in (3.4) is elliptic. The radii r of the circles of the u^2–lines satisfy $0 \leq r \leq \sqrt{c^2 - \lambda^2}$.

Figure 3 illustrates a parabolic, a hyperbolic and an elliptic pseudo–spherical surface.

Figure 4 illustrates a hyperbolic spherical and hyperbolic pseudo–spherical surface with the u^2–lines of minimal and maximal radii.
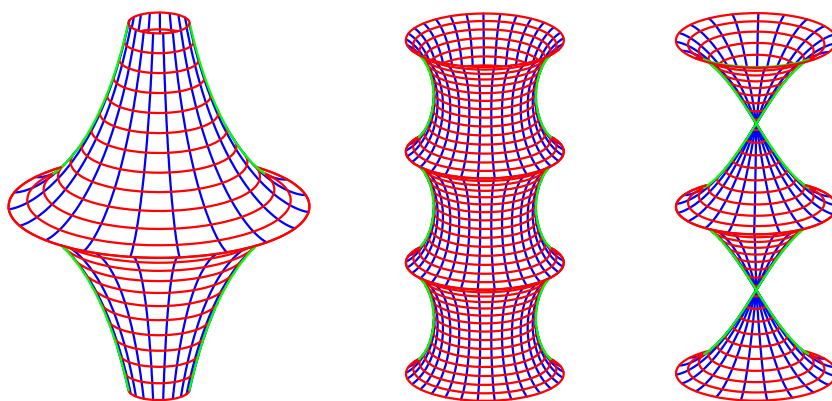
Figure 3.   Pseudo–spherical surfaces for $K = -1$ and $c = 1$
left: parabolic, $\lambda = \dfrac{3}{2}$;   middle: hyperbolic, $\lambda = \dfrac{3}{4}$;   right: elliptic $\lambda = \dfrac{1}{2}$
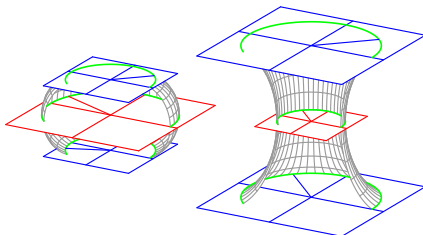

Figure 4.   A hyperbolic spherical and hyperbolic pseudo–spherical surface with the u^2–lines of minimal and maximal radii

4. The User Interface

Our software package includes a Windows application for the visualization of spherical and pseudo–spherical surfaces. The application allows a user to simply change a number of the parameters for the surfaces, without programming. The visualization is then done independently of any other software package.

The *Object window* appears in the Windows application, after the initial *introductory window*. Here the type of the surfaces and all their parameters can be chosen. After pressing the button *Draw*, the *Image window* appears and the drawing process automatically starts.

The Object window is organized in the *page control* manner due to the large number of *controls* in it. The page control has a number of *tab sheets* organized by the subjects and roles they play in the application.

The tab sheets are

Parameters,	for choosing the surface type, K and λ,
Intervals,	for the intervals for u^1 and u^2,
Drawing,	for the resolution, background color and for the decision to the draw the u^1, u^2 or contour lines,
U1,	for the number, color, thickness, visibility and the parameters for the technique of drawing which is not subject of this paper (such as approximation of line or steps of interpolation),
U2,	the same as for the u^1 lines,
Contour,	the same as for u^1 lines,
World Interval,	for the 3D–box which the whole object is placed in; this can be done automatically or by the user,
Projection,	for the center of projection and the projection plane of the central projection,
Output,	for the types of output files,
Description,	for a short description of the parameters.

The tab sheet Parameters is for setting the type of the surface and the parameters K and λ. First we choose between spherical and pseudo-spherical surfaces. Inside the group, we choose whether we want to draw a sphere/parabolic, hyperbolic or elliptic surface.

Once we have chosen the surface type, the program automatically takes the default parameters. This is useul because it is not very easy to find the suitable parameters for a good result. The program automatically computes the interval for u^1 by applying the rules given in Section 3, and displays it in the tab sheets Parameters and Intervals. Changing the parameters later leads to a re-computation of the interval.

Although the interval for u^1 is made correctly, it may happen that it cannot be used to the very ends, due to the numerical approximations of the intermediate functions and computer limits. To overcome this problem, we introduced some corrections of the endpoints of the interval. The theoretical maximal and chosen interval boundaries are displayed.

The parameters can be saved in a file with the extension the MVG and loaded from it later.

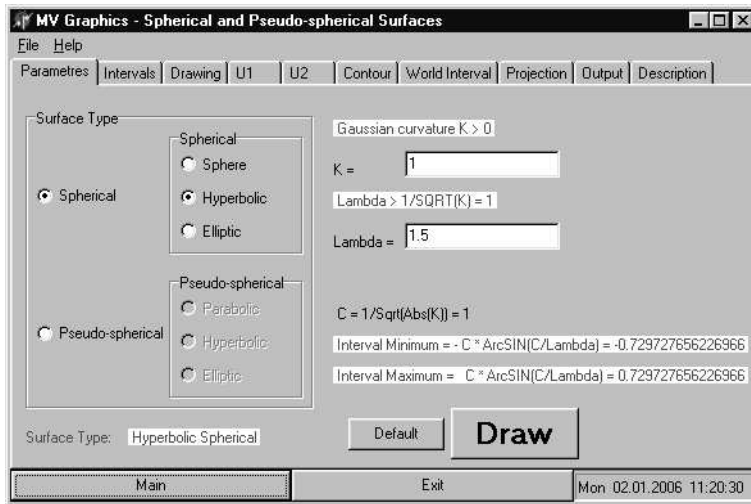Figures 5 and 6 illustrate the tab sheets Parameters and Intervals of the user interface of the Windows application.
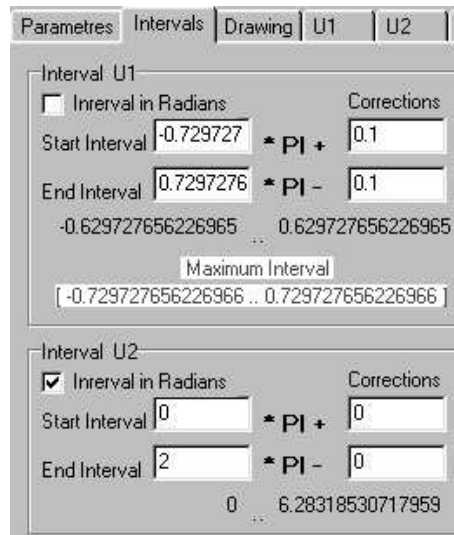
Figure 5.   A tab sheet Parameters

Figure 6.   Tab sheet Intervals

5. Conversions

Our figures can be exported to several formats such as BMP, PS, or PLT.
The images in these formats can be further converted to a number of other

formats by means of any image converter software, for instance, to a GIF file which can be included in an $HTML$ file, an EPS file which can be included in a TₑX or LᴬTₑX file, or a PNG or PDF file which can be included in a TₑX or LᴬTₑX file which is directly converted into a PDF file by means of PDFLᴬTₑX.

We use the software package Animagic GIF 32 to create an animation in animated GIF format from a sequence of our figures stored in GIF format, and include the animation as an animated GIF image in an HTML file.

6. Conclusions

The software package described in this paper can be a basis for the future work for producing the figures which can be part of TₑX or LᴬTₑX files. They can be published on the Internet as GIF or animated GIF files, or to illustrate certain properties of the surfaces of rotation with a given constant Gaussian curvature.

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