STRUGGLING WITH VARIABLES, PARAMETERS, AND INDETERMINATE OBJECTS OR HOW TO GO INSANE IN MATHEMATICS

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Pursuing our investigation on students' use and understanding of algebraic notations, this paper examines students' cognitive difficulties related to the designation of an indeterminate but fixed object in the context of the generalization of patterns. Stressing the semiotic affinities and differences between unknowns, variables and parameters, we examine a Grade 11 mathematical activity in which the core of the students' relationship to algebraic formula comes to light. We show how the semiotic problem of indeterminacy reveals the frailty of students' understanding of algebraic formulas and how it puts into evidence the limited scope of the use of formulas as schemas, strongly rooted in student's relationship to algebra.

INTRODUCTION

Making sense of letters is one of the fundamental problems in the learning of algebra. A letter is a sign, something that designates something else. In the generalization of patterns, letters such as '*x*' or '*n*' appear as designating particular objects –namely, variables. A variable is not a number in the arithmetic sense. A number, e.g. the number 3, does not vary. A variable is an algebraic object. Previous research has provided evidence concerning the meanings that students attribute to variables (e.g. MacGregor & Stacey, 1993; Trigueros & Ursini, 1999; Bednardz, Kieran, & Lee, 1996). One of these meanings consists in conceiving of a variable as an indeterminate number of a specific kind: it is not an indeterminate number *in its own*. For many students, it is merely a *temporally indeterminate number* whose fate is to become determinate at a certain point. Aristotle would have said that for the students, variables are often seen as "*potentially* determinate" numbers, as opposed to the numbers in the elementary arithmetic of our Primary school (e.g. 1, 4, 2/3 and so on), which are "*actual* numbers". Yet, the algebraic object "variable" should not be confounded with another algebraic object –the "unknown" (Schoenfeld & Arcavi, 1988; Radford, 1996). Although both are not-known numbers and, from a symbolic viewpoint, the same syntactic operations can be carried out on them, their meaning is different. In the algebraic equations used in introductory algebra, such as '$x+12 = 2x+3$', the unknown exists only as the designation of a number whose identity will be disclosed at the end. The disclosing of the unknown's identity is, in fact, the aim of solving an equation. In contrast, when '*n*' refers to a variable (see e.g. the pattern in Fig 1 below), the focus of attention is not on *finding* actual numbers but on the variable *as such*. The same holds for the expression '$2n+1$', that designates the variable "the number of toothpicks in figure 'number *n*'" (see Fig 1 below). In other words, in equations, we go from symbols (alphanumeric expressions) to numbers, while in patterns we go the other way around (of course, once established, a formulaic expression of a variable like '2*n*+1' can be used to find out specific values of '*n*' or '2*n*+1'). What previous research has suggested is that, for many students, letters (such as '*n*' in '2*n*+1') are considered as potential numbers −indeterminate ones waiting in a kind of limbo for their indeterminacy to come to its end. The letter is hence, for the students, an *index* (Radford, 2003), a sign that is indicating the place that an actual number will occupy in a *process* (Sfard, 1991) temporarily in abeyance (we shall come back to this point later).

In this article, we pursue our investigations of students' algebraic thinking about variables. We are interested in understanding the way that students cope with another algebraic object: a parameter, that is, an indeterminate but *fixed* element of the "values taken" by a variable. The paradoxical epistemic nature of this algebraic object rests on its apparent contradiction: it is a fixed, particular number, yet it remains indeterminate in that it is not an actual number. Like the variable from where it emerges, it is indeterminate and is not subjected to an inquisitorial procedure that would reveal (as is the case with unknowns) its hidden numeric identity. From an education viewpoint, the question is: How can such an object become an object of thought for the students? Because of its indeterminate and abstract nature this object cannot be pointed out through a gesture as we can point e.g. to one of the first terms in a given patter (see e.g., the pattern below; Fig 1). The only way that a parameter can become an object of thought is through the interplay of various sorts of signs. The next section provides some details about how we introduced this object in the course of a regular classroom mathematics lesson about patterns. The rest of the article is devoted to the analysis of some of the students' difficulties in making sense of a parameter.

METHODOLOGY

Data collection: The paper reports parts of a five-year longitudinal classroom research program where teaching sequences were elaborated with the teachers. The research involved four northern-Ontarian classes of grade 11, from two different schools. The same methodology was applied in both schools: the classes were divided into small groups of three to encourage students to work together and share their ideas with the others members of the group; then the teacher conducted a general discussion allowing the students to expose, confront and discuss their different solutions. During the implementation of the teaching sequences in the classroom, both the teacher and the researcher were present, willing to answer the students queries as they solved the problem. In each class, three groups were video-taped, the dialogues transcribed and written material was also collected. For the purpose of the present article, we will closely focus on one group we found representative of most students' work. This group was formed by Denise, Daniel and Sam.

About the given task: The teaching sequence included three linked problems concerning the construction of geometric-numerical patterns. The figures that constituted the patterns were described as being composed of toothpicks, triangularly disposed.

In the **first problem** the first figures of the pattern (also called "original" in the subsequent problems) were drawn (see Fig 1). After having been asked to find out the number of toothpicks for specific figures, the students had to write an algebraic formula to calculate the number of toothpicks in figure 'number *n*'.

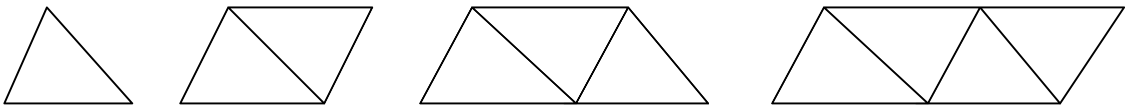

Figure 1 Figure 2 Figure 3 Figure 4

Fig 1: Original pattern.

The pattern in the **second problem** was related to a fictitious character (Mireille) who was said to have begun her pattern at the fourth "spot" of the original pattern (*à la place numéro quatre*, in French). The first figures of the pattern were also provided (see Fig 2) and the questions were similar to those of the first problem.

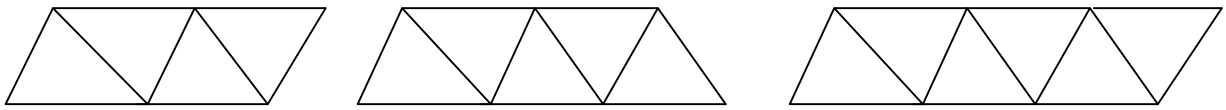

Figure 1 Figure 2 Figure 3

Fig 2: Mireille's pattern.

In order to investigate the students' cognitive difficulties in dealing with parameters, a new pattern (Shawn's) was introduced in the **fourth problem.** The spot where the pattern began was given, yet not specified: students were told that Shawn had begun his pattern at the "spot *m*" of the original pattern. They were then asked to provide an algebraic formula, in terms of *m*, that indicates the number of toothpicks in figure number 1 of Shawn's pattern. In what follows, we will focus on the fourth problem. Special attention will be given, however, to students prior answers, for it provides essential information about the students' relationship to algebraic symbols and, in particular, to their use and understanding of letters.

STUDENTS' RESPONSES

The semiotic problem of multiple referents

Both the first and second problem were easily solved. Thus, in the first problem, right after a quick numerical examination of the link between the number of the first figures and their corresponding amount of toothpicks, the students rapidly worded the description of a sequence of numeric actions: Denise said: "So it's times 2 plus 1, right?" and, to calculate the number of toothpicks in figure number, 25 effected the calculation $25 \times 2+1$.

Denise's utterance is the description of a numeric schema (in Piaget's sense) that allowed the students to obtain the formula by translating it into symbols. However, when translating the worded schema into an algebraic formula, students produced a response attesting some lack of precision in the meaning that they gave to symbols:

x = # figure

$n = 2x + 1$

Fig 3: Students' answer to the last question of Problem 1.

Indeed, the translation of the worded schema of the students' response suggests that they do not interpret the letter '*n*' as standing for the number of an unspecified figure (despite the fact this had been suggested in the text). Instead, by writing "$n = 2x+1$" (see line 2 Fig 3), '*n*' designates an amount of toothpicks. Furthermore, a new letter was introduced to designate something that remained *implicit* at the verbal level but which was nonetheless substituted by actual numbers (such as 25, to answer the question of the number of toothpicks in Figure 25). The letter '*x*' used by the students to designate the number of the figure (see line 1 Fig 3) plays the role of *index*, i.e. something indicating a place that will be occupied by a number. The letter '*x*' designates a "temporarily indeterminate" number, suffering from indeterminacy, seen as a kind of sickness that, like a cold, should sooner or later come to its end. The letter '*n*' designates the schema '$2x+1$'. Instead of considering '*n*' as a genuine algebraic variable, the transcripts and video analyses of this and other groups suggest that '*n*' is seen as a "potentially determinate" number, a number that will become "actual" (in the Aristotelian sense) as soon as '*x*' takes on its numerical value.

Bearing these antecedents in mind, let us now turn to the forth problem, where the students encountered the concept of *parameter*. Imagining the letter '*m*' as an indeterminate yet fixed number at the starting point of a new pattern posed many difficulties to them:

1.1 Daniel: OK, but if it begins at spot number *m*, and we want to know figure number 1, isn't this 1? Isn't *m* [equal to] 1?[...]
1.2 Daniel: But isn't *m* the number of the figure?
1.3 Denise: It's... the place where...
1.4 Daniel: OK, it's not... OK, it's a number of figure, but, OK…

The above excerpt illustrates some of the fundamental student difficulties in trying to make sense of the question. In order to understand these difficulties, we need to discuss three different ways of referring to the figures. In the previous problems, indeed, the figures can be seen from different perspectives:

Figure as substance: Each figure can be referred to through the number of toothpicks it is made of. For instance, in the original pattern, there is one 3-toothpick figure, one 9-toothpick figure (namely Figure 1 and Figure 4 respectively). In

Mireille's pattern, there are no figures with 3 toothpicks, but there is one 9-toothpick and one 11-toothpick figures (Figure 1 and Figure 2, respectively).

Names as part of a system: Each figure can also be referred to by a "label". This label is its "name" (Figure 1, Figure 2, etc.). This name corresponds to the *relative* position among the others figures *of the same pattern*. For instance, in the original pattern, as well as in Mireille's pattern, "Figure 1" is the label of the first figure, "Figure 2" of the second figure, etc.

Relativeness of the object's name: Since Mireille's and Shawn's patterns begin at a given place or "spot" in the "original" pattern, the *place* that each figure occupies inside a certain sequence must be distinguished from the place these figures occupy in the original pattern. For instance, the figure called "Figure 2" in Mireille's pattern is called "Figure 5" in the original pattern.

Line 1.1 is representative of the difficulty in seeing the subtle relativeness of the object's name. Indeed, from the point of view of the original pattern, Shawn's Figure 1 is at the spot '*m*'. But each first figure in a pattern starts at spot 1 of its own pattern. By saying that '*m*' is 1, Daniel, probably uncomfortable with the indeterminacy, merges the two referents.

Besides being related to a place in the pattern, '*m*' also corresponds to the number of the figure that occupies this place. In this sense, in Line 1.2 Daniel was right when saying that '*m*' corresponds to the number of a figure: if we consider the original pattern as reference, '*m*' is indeed the number of the figure. In Shawn's pattern, however, this figure is no longer "Figure *m*": it becomes "Figure 1".

The effect of the indeterminate origin on using a schematic formula

As we saw previously, the students rapidly came up with a formulaic schema for the number of toothpicks of a figure located at an *indeterminate* place –namely, '*n*' (see Fig 3). The formulaic schema made sense for the students insofar as it was considered as a process in abeyance. Now, how were they to find an algebraic expression for the number of toothpicks in a figure for which the place ("m") was no longer to be considered temporally indeterminate but *indeterminate as such*? Noticing the students' struggle to make sense of the question and their reaching an impasse, the researcher went to talk to the group:

2.1 Researcher:	They ask you to find an algebraic expression, in terms of 'm', that indicates the amount of toothpicks that there are in the new pattern. It starts at figure 'm' […] How many toothpicks will its first figure have?
2.2 Sam:	Yeah, well we don't know this.
2.3 Daniel:	Well, that's what we have to find out. [...]
2.4 Daniel:	His 1, his 1, where is it located according to this (pointing the "original" pattern) . (...) Where is the 'm' according to this? [...]
2.5 Denise:	So, if you want to find the amount of toothpicks in his pattern (*sic*), if you had the number of the figure you could do it, but we don't have it. That's the only thing I don't know how to do.

The difficulty of conceiving of the indeterminacy of the spot '*m*', not as a temporal indeterminacy but as indeterminacy *as such*, checkmated the students' formulaic schema (see lines 2.3, 2.4 and 2.5). The students needed to understand that a parameter is an indeterminate but fixed element of the "values taken" by the variable and that despite its indeterminacy it makes sense to think about it and of the figure at that place, even if no numerical value can be attributed to them. Understanding this entails understanding that there is a new layer of mathematical generality, a layer where the "existence" of the objects does not depend on numerical determinacy, whether actual or potential. The fact that this indeterminacy directly concerned the first figure of the pattern –its origin– was for the students, to say the least, most disconcerting:

3.1 Daniel: We don't have Shawn's pattern. [...] We don't know where it starts at and where it ends... We can almost not do it […]

3.2 Sam: I'm going insane.[...] We have nothing...

Acceptance of indeterminacy is a real obstacle to the students. As their dialogue indicates, they seem to feel the need to attribute a numerical value in order to progress in the mathematical activity. This particularity reveals the students' understanding and use of letters in algebraic formulas, suggested elsewhere in their answers for prior problems. For them, even though they are able to produce a formula and manipulate it (e.g. substituting), the formula is still seen as a process and not yet as an object (Sfard, 1991). In other words, we might say that they accept dealing with formulas, dealing with the indeterminacy, but only for a while, for the formulas have to provide a result:

4.1 Daniel: We just don't know how to find '*m*'. [...] What did you say?

4.2 Denise: $x = 2m+1$.

4.3 Researcher: Do you agree with that? [...]

4.4 Sam: Yeah, but it takes you nowhere. It's nice to have a formula, but you have to get a number.

4.5 Researcher: We don't have to have a number!

4.6 Denise: We have nothing.

The students focused on trying to determine *m* and, by analogy with previous problems, they struggled to provide a formula that, at the end, would give an output for *m*. But what exactly is *m* for them? How did the students express it in a formula?

Among all the referents that characterize the figures, there is one to which the students granted a privilege: the number of toothpicks that a figure is made up of (influenced maybe by the questions in the first problems that focused on finding the number of toothpicks in particular figures or on finding a formula that would generalize this amount). As students progressed in solving the problems, their associating of the figure with its number of toothpicks became, indeed, more and more evident:

5.1 Researcher: What spot did Mireille start at?

5.2 Denise and Daniel: At 9.

Not surprisingly, the first attempt in interpreting '*m*' was hence to consider it as representing the number of toothpicks and, by analogy with the formula that they provided in the first problem ('$n = 2x+1$'), Denise suggested the formula '$m = 2x+1$' for the first question of Shawn's problem: "(...) It's the same formula as this one (*pointing at the formula '$n = 2x+1$'*) (...) So $m = 2x+1$?". As previously mentioned, in the students' response to the first problem, '*n*' designated the number of toothpicks of the figure 'number *n*'. When Denise proceeded by analogy to solve the problem 4, this would mean that she considered '*m*' for the number of toothpicks in figure '*m*'. But the question was to provide a formula that would indicate the number of toothpicks in *figure number 1* of Shawn's pattern. Because of their merging of the multiple referents and, more precisely, because of the confusion between the place where Shawn's pattern began (in the original pattern –that is, place *m*) and the name of the related figure in Shawn's pattern (that is, of its first figure –Figure 1), the formula '$m = 2x+1$' stands for the number of toothpicks in figure 1 of Shawn's pattern. But Denise feels uncomfortable with the formula that she has just provided and says: "That's strange, they say how many toothpicks there will be in figure 1 of Shawn's pattern, but we don't have *x*." Notice that Denise has transformed the original question into a different one: finding an algebraic expression has been "translated" into finding an amount of toothpicks. What Denise finds strange is that one could ask such a question without providing her with an *actual* number.

It is only after having realized the difference between the multiple referents –and only then− that Denise is able to provide the expected formula: "$x = 2m+1$, because if '*m*' is the place where he starts his pattern at, that's still not figure number 1, oh, yeah!". Yet, because the inquisitorial procedure is strongly rooted in students' conceptions of formulas, they do not find this answer acceptable:

6.1 Daniel: Yeah, this would work, yeah, it's just *m* that we don't know how to find.

6.2 Denise: We don't know how to find it. Yeah, that's the thing.

CONCLUDING REMARKS

The mathematical activity reported in this paper suggests a context in which the core of students' understanding of letters and their conception of formulas comes to light. When considering the ease with which the students solved the first problems, one may be tempted to conclude that the students have successfully conceptualized letters as variables and have been able to meaningfully produce and even manipulate formulas. Indeed, the students were perfectly at ease dealing with the concept of 'figure *n*' −a concept that posed great difficulties to them and that took time to overcome when first introduced in Grade 8 (see Radford, 2000). Yet, the semiotic problem of indeterminacy brought forward by the concept of *parameter* in problem 4 reveals the frailty of students' understanding of algebraic formulas that the answers provided in first problems hide. In particular, it highlights the frailty of perceiving formulas as schemas, putting into evidence their limited scope. It required a different situation –one demanding the students to deal with a new level of generality– to reveal the students' difficulties. In the context of the generalization of patterns, this means making the students consider the figures not as necessarily characterized by actual or potential numbers but as genuine conceptual objects, objects that can only be referred to through signs. Perhaps the philosopher Immanuel Kant was right in asserting that the possibility of (elementary) geometry resides in our intuition of space and that the same cannot be said of the objects of algebra, whose possibility cannot even be attributed to our intuition of time. Their possibility resides in symbols. From an education viewpoint, our results suggest that a pedagogical effort has to be made in order to make the students understand that there is layer of generality in which mathematical objects can only be referred to symbolically, detached in a significant manner from space and time. The students need to learn to cope with the kind of indeterminacy that constitutes a central element of the concepts of variable and parameter. Although one may very well be asked to begin from "nothing" (see Sam in passage 3.2) there is no reason to go insane: one still can go somewhere else –to symbolic algebra.

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