A CASE STUDY OF HOW KINESTHETIC EXPERIENCES CAN PARTICIPATE IN AND TRANSFER TO WORK WITH EQUATIONS

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The broad goal of this report is to describe a form of knowing *and a* way of participating *in mathematics learning that contribute to and further alternative views of transfer of learning. We selected an episode with an undergraduate student engaged in a number of different tasks involving a physical tool called "water wheel". The embodied cognition literature is rich with connections between kinesthetic activity and how people qualitatively understand and interpret graphs of motion. However, studies that examine the interplay between kinesthetic activities and work with equations and other algebraic expressions are mostly absent. We show through this episode that kinesthetic experience can transfer or generalize to the building and interpretation of formal, highly symbolic mathematical expressions.*

INTRODUCTION

How experiences and knowledge from one situation transfer or generalize to another situation has long been a topic of interest (e.g., Thorndike, 1906; Judd, 1908; Wertheimer, 1959). In recent decades researchers have posed alternatives to what now is commonly referred to as a classical or traditional view of transfer (Lobato, 2003; Tuomi-Grohn & Engestrom, 2003). Many of these alternatives are grounded in situated and socioconstructivists perspectives rather than in behaviorist or information processing perspectives. For example, Hatano and Greeno (1999) argue that rather than treating knowledge as a static property of individuals that is correctly or incorrectly applied to new tasks (which is compatible with traditional views of transfer), more emphasis should be placed on the norms, practices, and social and material interactions that afford the dynamic and productive generalization of learning. Hatano and Greeno further argue that alternative views of transfer offer researchers insights into how "students may develop quite different *forms of knowing* when they learn in practices that involve *different ways of participating*" (p. 650, emphasis added).

The broad goal of this report is to bring together a different *form of knowing* with a different *way of participating* in mathematics learning and in so doing contribute to and further alternative views of transfer. Classic forms of knowing include knowing-how and knowing-that (Ryle, 1949). These forms of knowing tend to be static, purely mental, and compatible with traditional views of transfer that look for direct application of knowledge. A different distinction in forms of knowing that is potentially more useful for alternative views of transfer is that of knowing-with and knowing-without. Knowing-with characterizes aspects of meaning making as it

relates to developing expertise with tools. *Knowing* a mathematical idea *with* a tool, for example, (1) engages multiple and different combinations of dwelling in the tool, (2) invokes the emergence of insights and feelings that are unlikely to be fully experienced in other ways, and (3) is in the moment. The opposite of knowing-with is knowing-without. We all have had experiences of knowing-without embedded in feelings of something being alien, foreign, and belonging to others. The difference between knowing-with and without is not absolute but contextual (Rasmussen & Nemirovsky, 2003; Rasmussen, Nemirovsky, Olszewski, Dost, & Johnson, in press). These characteristics of knowing-with resonate with many of the features of Lobato's (2003) actor-oriented perspective on transfer and Greeno, Smith, and Moore's (1993) situated view of transfer.

In addition to different forms of knowing, Hatano and Greeno (1999) direct our attention to different ways *of participating* in mathematics learning. In this work we draw on recent advances in embodied cognition that highlight the centrality and significance of learners' gestures and other ways of kinesthetically participating in mathematical ideas. Nemirovsky's (2003) review of embodied cognition distills two conjectures regarding the relationship between kinesthetic activity and understanding mathematics that help frame this research report. First, mathematical abstractions grow to a large extent out of bodily activities. Second, understanding and thinking are perceptuo-motor activities that are distributed across different areas of perception and motor action. We also note that the embodied cognition literature is rich with connections between kinesthetic activity and how people qualitatively understand and interpret graphs and motion (e.g., Nemirovsky, Tierney, & Wright, 1998; Ochs, Jacobs, & Gonzales, 1994). It is noteworthy, however, the absence of studies that examine the interplay between kinesthetic activities and work with equations and other symbolic expressions. Thus, the focused goal of this report is to investigate the ways in which kinesthetic activity can participate and transfer to work with conventionally expressed equations.

LITERATURE REVIEW ON TRANSFER

At the beginning of the century Thorndike (Thorndike, 1906; Thorndike & Woodworth, 1901) conducted the first series of "transfer studies." Since then, the overall scheme of these studies became established: subjects who have had experience with a source or learning task are asked to solve a target or transfer task, and their performance is compared to a control group. In looking back at the many studies and debates on the notion of transfer of learning that were developed during the twentieth century, we will describe what we recognize as dominant themes and concerns in the literature.

The aim of most of the transfer research has been to predict and identify the conditions under which transfer does or does not happen. On the one hand we intuitively know that in everyday life we are constantly "transferring" in the broad sense; that is, we are making connections to our past experience, bringing metaphors

to life, sensing a stream of thoughts populated by unexpected associations, and so forth. On the other hand, the results of transfer research have led many researchers to conclude that transfer is rare and difficult to achieve unless it is "near" or based on source and target situations that are markedly similar (Singley and Anderson, 1989). This mismatch between common expectations and the results of the transfer studies is, to this day (Anderson, Reder, & Simon, 1996; Lave, 1988), a centerpiece of the debates.

In order to predict the occurrence of transfer and to conduct empirical corroboration, theorists postulated several different types of transfer mechanisms. These mechanisms have centered on the preservation of structures, that is, on the thesis that transfer takes place when certain structures present in the subject dealing with the source task are re-activated when dealing with the target tasks. Thorndike proposed that what one learns in a certain domain transfers to another domain only to the extent that the two domains share "identical elements."

On the other hand, during the period dominated by information-processing approaches, the preservation of mental structures came to be seen as the key for the occurrence of transfer. The idea was that, rather than the features of the tasks themselves, what matters is how people conceptualize the tasks; in other words, the mental structures that subjects bring to bear when they deal with the tasks (Singley & Anderson, 1989).

Transfer studies often cite the literature on "street mathematics" which examined the ways in which people in different cultures solve arithmetic problems from everyday life (e.g., Lave, 1988; Nunes, Schliemann, and Carraher, 1993; Saxe, 1982). We think these studies question the idea that there are some mathematical procedures that are optimal for everyone at all times. This research has repeatedly shown that people compose solutions to the problems they face by combining multiple approaches as well as the resources and demands of the situation at hand. There is nothing exotic about creating idiosyncratic procedures and merging practices, on the contrary it is common and widespread.

As new teaching practices inviting students to invent algorithms are becoming part of schooling, it is increasingly clear that the diversity of approaches and dynamic composition of solutions can be as typical in the school as it is in the street. The old idea that there are some mathematical procedures that are optimal for everyone at all times is an artifact of cultural practices traditionally associated with schooling. The main issue made prominent by research on street mathematics is not, we believe, that school-based algorithms fail to transfer, but that people, rather than using pieces of knowledge as ready-made structures that get applied to new situations, compose solutions by making use of multiple approaches and tuning them to the resources and demands at hand. In this report we examine how prior kinesthetic experiences with a physical tool can offer students resources that can be generalized to work with symbolic equations.

METHOD

We conducted a total of eight, 90- to 120-minute open-ended individual interviews with three students. In the interviews students engaged in a number of different tasks involving a physical tool called the water wheel. As shown in Figure 1, the water wheel consists of a circular plexiglass plate with 32 one-inch diameter plastic tubes around its edge. Each tube has a small hole at the bottom. The plate turns on an axle and is free to rotate. The tilt of the axle can be adjusted between 0 and 45 degrees from vertical. Water showers into the eight uppermost tubes from a curved pipe with holes along its underside.

A submersible pump sends water to the pipe, with a valve to regulate the flow. An oil bath between nested cylinders provides dynamic friction for the axis of rotation. Raising or lowering an oil reservoir varies the oil level in the cylinders. The angular velocity of the water wheel is measured by two photogates that detect the motion of a pattern of black lines on the wheel top.
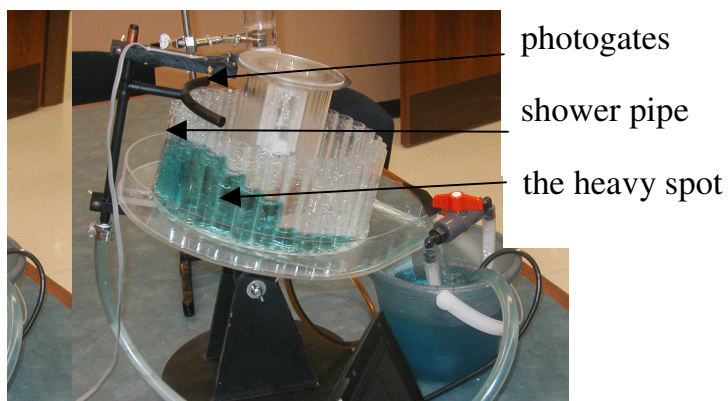

Figure 1. The water wheel

A computer interface permits users to graph angular velocity versus time, angular acceleration versus time, and angular velocity versus angular acceleration while the wheel is turning (Nemirovsky & Tinker 1993). Water showers into the tubes when they are carried underneath the shower pipe. As the wheel turns, the water gathered in each tube provides a torque around the axis of the wheel. Because each tube leaks water from the bottom, the amount of water in each tube decreases over time, until that tube again swings upward to the shower pipe to receive more water. With different choices of tilt angle, flow rate, bearing friction, and initial water distribution, the motion of the wheel exhibits a variety of periodic, almost periodic and chaotic motions, as well as period doubling and transitions into chaos. During periodic motion, water tends to accumulate in a bell-shaped distribution in the tubes, which students often call "the heavy spot" (see Figure 1).

Touching and sensing the heavy spot was a critical and significant experience for students. For example, in the second interview "Jake" predicted that a certain graph of velocity versus acceleration would be circular in shape. Computer generated graphs of actual data, however, indicated the graph to be dimpled on the top and bottom, like an apple. Jake ultimately concluded that the apple shape had to be the case by physically touching and sensing the forces at play in the motion of the wheel (see Rasmussen & Nemirovsky, 2003 for more detail).

Each student we interviewed had completed three semesters of calculus and had taken or was taking differential equations. The interviews used a set of preplanned

tasks as a springboard for exploration of mathematical ideas that were of interest to the student, rather than as a strict progression of problems to complete. We also actively worked in the interviews to establish an environment in which the student felt comfortable exploring new ideas and explaining their thinking, however tentative. All interviews were videotaped and transcribed. Summaries of each interview were developed and compared across all interviews. In this report we focus on the learning of one student, Jake, in his third and final interview because it was most helpful in our understanding how kinesthetic activity with a tool can transfer to work with symbolic equations.

MATHEMATICAL IDEAS INVESTIGATED

The first two interviews focused on qualitative and graphical interpretations of motion while the third interview, which is the source of data for this paper, focused on interpretation of the system of differential equations that model the motion of the water wheel.

We planned for students to engage in reasoning about a variety of different phase plane representations. A typical example of a phase plane is the R-F plane for a system of two differential equations dR/dt and dF/dt, which might, for example, model the evolution of two interacting populations of animals such as rabbits R and foxes F. For instance, consider the system of differential equations, $dR/dt = 0.2R - RF$, $dF/dt = -F + 0.8RF$, intended to model the population of rabbits and foxes. Students in modern approaches to differential equations are often required to interpret the meaning of the individual terms in the equations. For example, why is it the case that the first equation has a minus RF term while the second equation has plus RF term? Students in these interviews had engaged in similar analyses in their differential equations course for equations like dR/dt and dF/dt and had developed a number of interpretive strategies. One strategy was to view the RF terms as an indication of what happens to the populations when the two species interact. Another strategy was to interpret the equations when either R or F is zero. An information processing approach would judge successful (or not) transfer in terms of the extent to which these interpretive strategies were employed in the novel task with the water wheel.

The phase plane analyses that we planned to use with the water wheel centered on graphs in the angular velocity-angular acceleration plane, coordinated with time series graphs, and with the motion of wheel. In the third interview we invited students to engage in interpretive analyses of the following system of three differential equations that model the motion of the water wheel: $dX/dt = \sigma(Y - X)$, $dY/dt = -Y + XZ$, $dZ/dt = R - Z - XY$. The variables X, Y, and Z are dimensionless combinations of physical variables, each with a fundamental meaning. X represents angular velocity, Y represents the left-half right-half water imbalance, and Z represents the top-half bottom-half water imbalance. If more water is in the right half of the wheel, then Y is positive. A negative value for Y indicates that more water is in the left half

of the wheel (such as the instant in time shown in Figure 1). Similarly, a positive Z value means that more water is located in the top half of the wheel. The parameter R essentially relates to the pump flow rate and tilt while the parameter σ relates to the amount of friction (oil level). All of this was explained to the students in the interview.

The -Y term in the equation dY/dt = -Y+XZ accounts for the fact that water flows out of the tubes in such a way that any differences in their left-right distribution tend to nullify. Both sides tend to have less and similar amounts of water. This happens faster if Y is bigger. From this perspective, dY/dt might be understood as the rate at which the left-right imbalance is evening out. Jake's *knowing* Y *with* the heavy spot cultivated a different perspective on dY/dt. As we elaborate in the next section, Jake's earlier kinesthetic engagement with the water wheel's "heavy spot" afforded him novel and productive ways to make sense of various terms in the differential equations.

ANALYSIS AND DISCUSSION

We often see students designing graphs to produce narratives of perceptuo-motor events, but the use of standard symbolic notations often seems less likely to elicit such direct unfolding of interpretation. An important contribution we make in this report is to clarify and document that kinesthetic experiences can play the role of "bridges" that experientially bring together partial results obtained by symbol manipulation with certain "states of affairs" that students have engaged with physically. In the following example, which is typical of Jake's work with the equations, kinesthetic experiences anchor his interpretations of why the different terms in the differential equations make sense (or not).

His analysis of the differential equation dX/dt = σ(Y–X) began with an attempt to interpret the right hand side of the first equation as a whole. He reasons out what happens to the angular acceleration (since that is what he understands dX/dt to mean) when the amount of damping increases. As Jake worked through this approach, he began to tease out how the individual terms in the right hand side of the first equation might make sense to him. To do so, he returned to the idea of a "heavy spot," which he had introduced in earlier investigations, mainly of periodic motion. In this way, anchored in a special case, he built interpretations that will hold in general. The following excerpt picks up this conversation with Jake reflecting on whether it makes sense for the equation to include a positive Y term rather than a negative one (–Y).

> Jake: OK. Now, the positive term of Y, at least, uh, seem to make sense because, if the [holds his hands out, palms up], if it's the more imbalanced [Chris draws a circle on the board next to the equations], the, uh, more [makes a half rotation gesture], uh, the higher the acceleration. Because, if it's much more heavier on this side [cups hand over right side of the circle diagram on the board] than this side [cups hand over left side of the circle], then it seems to make sense that the pull due to this much heavier side [cups right side of circle]. Seems to be, uh,

much stronger and, therefore, it [gestures with a grabbing and pulling motion downward] seems to accelerate, uh, much more faster.

Chris: Mmmm. So, that's when Y is positive. [Jake: Right]. How about when Y is negative?

Jake: OK, yes. That's what I was going refer to. Um. Y, Y is negative in a situation where the, uh, uh, the heavier side is, on this side [points to left side of the circle]. And, um, and, if there's. So, the pull is this way [gestures down], therefore, the acceleration is negative [gestures in a counterclockwise swirling motion] instead of positive [gestures in a clockwise swirling motion].

As Jake began his explanation, Chris drew a circle on the board next to the differential equations. Jake's gestures (noted in the transcript) transform this circle into a diagram of the water wheel, with a heavy spot implicitly in evidence. For example, Jake cups a portion of this circle with his hand, as if he were grasping for the heavy spot. Jake's gesture, cupping his hand as if he had taken hold of the heavy spot, suggests a form of being the wheel, in the sense that forces and rotational movement are brought forth through the way he works with the circle diagram of the water wheel drawn on the board. In this way, his physical experience, interpreted through his concept of the "heavy spot," anchors his interpretation of the first equation. In a similar way, his physical experience, combined with key ideas that he has built in order to reflect on that experience, help Jake make sense of the remaining two equations. Other examples will be rendered in the presentation of this paper.

FINAL REMARKS

Representations, such as equations and graphs, are indispensable for mathematical thinking and expression. It is one thing to know, for example, that the slope of the graph of a certain function obeys a certain equation, while it is another thing to sense bodily the need to slow down and the different ways of slowing down. While these aspects can be dissociated, and in fact they often are (e.g., solving an equation without any kinesthetic sense of the motion it describes), they can be related in manifold and complex ways. It is possible that this widespread dissociation leads students to uncritically accept mistaken results obtained through formal calculation, because the latter tends to be performed without the guidance of intuitive expectations. In this report we showed that kinesthetic experience can transfer or generalize to the building and interpretation of formal, highly symbolic mathematical expressions. This existence proof has the potential to open new ground for research on embodied cognition and transfer.

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