DRAWING AS PROBLEM-SOLVING: YOUNG CHILDREN'S MATHEMATICAL REASONING THROUGH PICTURES

Carole Saundry and Cynthia Nicol

University of British Columbia, Canada

Our project investigates how young children think through mathematical problem-solving. The project examines how children respond when presented with a mathematical problem to solve, the kinds of pictures they draw spontaneously, the things they are thinking while they draw, and how teachers can support children in developing these skills in order to become better mathematical problem-solvers.Video footage and student work informs the creation of a framework for examining students' drawing as problem-solving and provide a lens for understanding children's mathematical reasoning as expressed through and derived from pictures.

RATIONALE

Mathematical problem solving involves a series of complex processes – identifying the problem, interpreting what is to be done, selecting and applying a strategy for solving the problem and then assessing the reasonableness of the solution. Teachers seek to support their students, particularly in the selecting and applying of strategies for problem solving, by offering a range of possible methods. Often this list of problem-solving strategies includes “draw a picture”.

Some children draw elaborate pictures for even the simplest of problems, representing their final solution in detailed, pictorial form. Students who focus on drawing the surface elements of problem (like the eyelashes of the people sharing the seats on the bus) might miss the mathematical point of the problem.

Other children use drawing in an entirely different way – with a different purpose and outcome. These children employ drawing “as” problem solving. For these children, the act of drawing is both a process and a product. In this case, drawing – or representing – is done during problem solving. For these children, mathematical representations, by virtue of their use as supports for thinking processes, are more iconic – representative of an idea, a numerical process or a mathematical concept.

Our project examines a child’s production of an image in response to a mathematical problem, and the nature and purpose of what children represent while they solve problems. This study focuses on drawing “as” problem solving, and through observation of a series of performance tasks with young children, outlines a framework for describing student’s representational thinking.

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 57-63. Prague: PME.

CURRENT RESEARCH

Wheatley and others (Cobb, Reynolds, Presmeg, Steffe; Thomas, Mulligan, and Goldin, 2002; Gray, Pitta, and Tall, 1999) have researched spatial sense and mental imagery, and have posited a connection between a child's capacity to make a mental image and his or her ability to solve mathematical problems. The ability to make and manipulate a mental image is seen to be key in problem-solving, leading a child to think more flexibly in response to a problem. Wheatley and Cobb state that

> "[m]athematical problem solving is often a matter of reasoning analytically, constructing an image, using the image to support additional conceptual reasoning… a process of building from images to analysis and analysis to images [that] may continue through many cycles." (1990, p. 161)

Researchers present varying perspectives on drawing in math class. Some maintain that drawing supports children in modeling a problem and therefore in arriving at a solution for it (Smith, 2003, Woleck, 2001). Woleck (2001) writes from a classroom-teacher/researcher perspective in her attempts to listen to her students' mathematical drawings. She describes her Grade 1 students' drawings in response to math problems and explains that these dynamic representations signify an essential cognitive bridge between the concrete and the abstract in mathematics. As children strip away the unnecessary features of a problem, removing all but the essential elements of the problem's structure, they become generalizable to more than the single context in which they are being used.

Smith's (2003) study of third graders' drawing while problem-solving, he noted that each child drew – both to manipulate the objects in the problem (drawing as problem-solving) and to represent their thinking afterwards (drawing of problem-solving). He noted that the children's use of drawings differed in terms of their idiosyncratic nature in relation to the problem context. Smith concluded that the artifacts children produce in solving mathematical problems – including language, drawings and constructions – cannot be considered separate from the students' mathematical reasoning.

Current Cyprian research focuses on the interpretation of an image and the degree to which the image supports or detracts from a child's capacity to problem-solve (Elia & Philippou, 2004). Focused largely on intermediate aged-children's interpretations of images, these researchers did not ask students to solve mathematical problems by drawing, but rather by reading information presented pictorially. In terms of image production, results indicated that intermediate-aged children were taught to use very specific schema for representing mathematical operations; however, the research did not address a child's spontaneous and constructed representations or the development of these images over time and with experience.

CONTEXT OF THE STUDY

Research in the area of young children's sense-making through the act of drawing is just developing. While there is acknowledgement that drawings assist children in

representing their mathematical thinking and that this form of representation has merit (Smith, 2003), there is little research focused on how representational capacity develops in young children, or what thinking transpires while drawing. Clearly, talk is central to understanding a child's internal representational capacity; this study uses an interview format to clarify and expand on what a child is thinking while he or she draws, and how the process of representing a problem supports the problem-solving process.

Several questions are raised: How do children represent actions (like sharing, joining or separating) in pictures? How do these paper-pencil images match what children see in their heads? How complex does a problem have to be before a child's internal representations must be supported by external ones?

RESEARCH METHODS

The participants for this study were 34 Grade 2 students from a suburban school district. An initial set of problems was given to all grade 2 students in small groups of 4-5 children at a time. This whole class data served two purposes – as a screen for potential individual participants, and as artifacts to inform the generation of a framework for considering drawing as problem-solving capacity. Following on from this whole-group experience, six individuals were selected for further in-depth interviews. The assessment of "picture use as a thinking scaffold" took place in a one-on-one interview format. Students in both settings were video-taped, both from above and using a hand-held camera to capture both the talk and the act of drawing as it happened.

Problems were both routine and non-routine in nature. Students were asked to solve a range of problems: a grouping and a sharing problem (e.g. 18 cookies. 12 children share. How many will each child get?) as well as a combination and two-step problem involving more than one operation (e.g. 18 wheels, how many toys could there be?). Children were presented with a piece of paper and a pencil, and the problems were read aloud to the students.

During the course of the child's problem-solving, the following metacognitive questions were asked: "Can you tell me what you're thinking? How do you know? What does this part of your drawing mean?" After the problems are completed, specific questions around the act of drawing were asked, including: "How did drawing help you think about the problem? How did drawing help you solve the problem?"

DATA ANALYSIS:

Data analysis involved repeated watchings of the video footage - both the footage gathered on the overhead stationary camera and that from the handheld camera used for close-ups. These two vantage points proved invaluable in the assessment of the students' processing of the problems through visualization. Careful observation of the footage of children working revealed a range of common responses to the problems

presented. Each response type (using a picture as a counter, using a drawing to keep track of or eliminate items) children's image-making efforts (lifting eyes up, thinking aloud about the pictures in their heads), and the degree of sophistication of the drawing itself (elaborate pictures, or more iconic representations) were annotated along with the time at which they occurred. In all, there were 15 different types of responses to the problems noted; as subsequent video was viewed and re-viewed, these 15 were grouped into four categories or drawing as problem-solving themes, namely: virtual manipulatives, systems, imagery and sophistication.

Students' comments - the questions they asked, their meta-cognitive talk, their interactions with other children - were transcribed verbatim. A further language sample was recorded when children were asked how drawing helped them solve the problem. These answers helped to clarify what students were thinking while they drew and the degree to which drawing was seen as helpful.

RESULTS

Video data of grade 2 students' processing of the sharing problem provided multiple indicators of drawing as problem-solving. Key themes emerged upon analysis of both the footage and the students' work in progress.

The use of pictures as manipulatives: Some children used their drawings as a kind of manipulative or counter. They manipulated their recorded images on the page, moved, eliminated, shared or divided pictures as a way of solving the sharing problem. Like using physical manipulatives, this particular use of pictures required movement; students represented the action of sharing of cookies or the grouping of wheels with lines, arrows or circles, and counted their representations as they would have if they had used physical counters.

Pictures as system support: For some, pictures or other iconic representations provided students with a scaffold for keeping track of the elements of the problem - a system for eliminating or distributing items, a way to systematically test possible solutions. For the sharing problem, (18 cookies, 12 kids), students using their pictures in this systematic way drew 18 cookies on one side of the page, and then people on the other side, crossing out cookies that they had "given away". In this way, children could apply a process of elimination. The picture itself was critical to keeping track of the elements of the problem, and children using their drawings in this way would often count, recount and check their partial solutions.

Sophistication of representation: The degree of sophistication of student drawings varied greatly; while some drew elaborate, artistic pictures, still others used simple iconic representations. While some children lost mathematical direction by focusing too heavily on elaborately detailed pictures of people and cookies, still others processed the mathematics through the act of drawing the "story" or problem situation. One child put himself right into the cookie-sharing situation and drew his own portion of the 18 cookies on a plate. His picture allowed him to represent - and really understand - his response of "we all get one cookie and a half of a cookie".

Imagery: Still other students did not draw a picture immediately, or even at all. On initial viewing of the hand-held video footage in which the camera's lens was focussed on pencil and paper, it appeared by their lack of drawing as though these students were confused by the problem or did not have a strategy for recording their ideas. Once the overhead camera footage was viewed, however, it was clear that these students were using a different kind of strategy for processing the problem: visualization. From a vantage point overhead, it was observed that these children spent time looking up and thinking, their eyes moving as though watching a movie in their heads. Several of these children recorded their solution to the problem by printing only a numerical answer on the page. One child spent more than 10 minutes without putting pencil to paper, eyes up and muttering to himself, then wrote "3 halfs" on his page. When asked how he had come up with the answer three halves, he stated simply that "it just came into my head". Without drawing, this child had still processed the information in the problem in a visual way.

Within each of these broad themes, individual indicators were recorded and analyzed by task and by working group in order to examine trends in the kind and type of drawing as problem-solving behaviors. Many children used more than one type of strategy for approaching the problem, moving from virtual manipulatives to visualization and then to the adoption of a system for processing the information. The most successful children in the solution of the problem were those who adopted and implemented a system (elimination, check and re-check), regardless of the sophistication of their drawing - or whether they drew at all.

Contribution to a current body of research

Some researchers seek to support students by moving beyond their constructed idiosyncratic drawings to more structured and formal representations. Novick, Hurley and Francis (1999) have defined 4 general-purpose diagrams that can be used to match the structure of any given problem; Diezmann and English (2001) describe ways that teachers can support their students in understanding the relationship between a problem and the diagram needed (2001, p. 86-87). Diezmann and English promote diagram literacy – “the ability to understand and use and to think and learn in terms of images” in their study of 10 year old mathematics students (p. 77). The authors’ desire to connect image-making and mathematical understanding is consistent with the work of Wheatley, Cobb and Yackel. An important distinction is the application of Diezmann’s research findings. While Diezmann and English highlight the importance of talk and modeling when working with diagrams, they also recommend that students be explicitly taught these four specific diagrams and when to apply them, and should practice them within structured problem sets. They maintain that students should be presented with sets of problems that can be represented with a particular diagram and that the relationship between a problem and its corresponding diagram should be explicitly taught. Although the participants in the Diezmann study are 10 years old and approaching a developmental stage at which these relationships could be introduced and discussed, caution must be exercised when accelerating this developmental process for younger students. The desire to

support diagram literacy (Nickerson, 1994) in elementary school-aged students should not supercede the natural development and constructed meaning-making for young children promoted by the NCTM and researchers like Woleck and Smith. Smith acknowledges that children need to bridge from idiosyncratic to mathematical representations in order to understand and communicate mathematics (p.273) but makes the case for sharing, discussing and analyzing peers' solutions as a developmentally appropriate way to bridge from one to the another.

CONCLUSION

To support teachers in understanding how it is that the act of drawing assists their students in problem-solving, and how drawing as problem-solving is often complemented by visualization, a framework for describing the indicators of thought through representation is needed. Providing early primary educators with ways to interpret students' performance when drawing to solve a problem will give them a tool to support their children in moving along the continuum. Exploring drawing as problem-solving may help teachers recognize and lend credibility to children's earliest representations, and help them to notice the process of problem-solving and not simply the product.

Research in the area of young children's drawing as problem solving is limited. The hope is that this study might contribute to a larger body of research related to the representation of mathematical thinking, and to fill a gap in considering how the process of drawing – not simply its product – has a central role to play in fostering children's mathematical reasoning. Our project seeks to address key questions related to this topic: complexity, problem-types, the relationship between problem language and student action, and by identifying observable indicators of mathematical reasoning while children are engaged in drawing, to contribute to the field of mathematical representation.

References

Diezmann, C. M. & English, L.D. (2001). Promoting the use of diagrams as tools for thinking. In A Couco (Ed.) *The roles of representation in school mathematics*. (pp. 215-227) Reston, VA: NCTM.

Gray, E., Pitta, D., & Tall, D. (1999, in press). *Objects, actions and images: A perspective on early number development.* Adapted version of a paper presented to the Elementary Number Research Forum of the XXI International Conference of the Group for the Psychology of Mathematics Education, Lahti, Finland.

Elia, I., & Philippou, G. (2004). The functions of pictures in problem-solving. In Johnsen Høines and Fuglestad (Eds) *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, Volume 2*. pp. 327-335.

Nickerson, R. S. (1994). The teaching of thinking and problem-solving. (pp. 409-449) In Sterberg, (Ed.) *Thinking and Problem Solving*. San Diego, CA: Academic Press.

Novick, L. R., Hurley, S. M. and Francis, M. D. (1999). Evidence for abstract, schematic knowledge of three spatial diagram representations. *Memory and Cognition* 27: 288-308.

Smith, S.P. (2003). Representation in school mathematics: Children's representations of problems. In Kilpatrick, J., Martin, W.G., & Schifter, Deborah (Eds.) *A Research Companion to Principles and Standards for School Mathematics*, (pp. 263-274). Reston, VA: NCTM.

Thomas, N. D., Mulligan, J.T., & Goldin, G.A. (2002). Children's representation and structural development of the counting sequence 1-100. *Journal of Mathematical Behavior*, 21, 117-133.

Wheatley, G. H. & Cobb, P. (1990). Analysis of young children's spatial constructions. In L. Steffe & T. Wood (Eds.) *Transforming Children's Mathematics Education: An International Perspective*. (pp. 161-173). Hillside, NJ: Erlbaum.

Woleck, K. R., (2001). Listen to their pictures: An investigation of children's mathematical drawings. In A Couco (Ed.) *The roles of representation in school mathematics*. (pp. 215-227) Reston, VA: NCTM.