Substitutions over infinite alphabet generating $(-\beta)$-integers

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1 Introduction

This contribution is devoted to the study of positional numeration systems with negative base introduced by Ito and Sadahiro in 2009, called $(-\beta)$-expansions. We give an admissibility criterion for more general case of $(-\beta)$-expansions and discuss the properties of the set of $(-\beta)$-integers, denoted by $\mathbb{Z}_{-\beta}$. We give a description of distances within $\mathbb{Z}_{-\beta}$ and show that this set can be coded by an infinite word over an infinite alphabet, which is a fixed point of a non-erasing non-trivial morphism.

2 Numeration with negative base

In 1957, Rényi introduced positional numeration system with positive real base $\beta > 1$ (see [7]).The β-expansion of $x \in [0,1)$ is defined as the digit string $d_\beta(x) = 0 \bullet x_1x_2x_3\cdots$, where

$$x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor \quad \text{and} \quad T_\beta(x) = \beta x - \lfloor \beta x \rfloor .$$

It holds that

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \cdots .$$

Note that this definition can be naturally extended so that any real number has a unique β-expansion, which is usually denoted $d_\beta(x) = x_kx_{k-1}\cdots x_1x_0 \bullet x_{-1}x_{-2}\cdots$, where $\bullet$, the fractional point, separates negative and non-negative powers of β. In analogy with standard integer base, the set $\mathbb{Z}_\beta$ of β-integers is defined as the set of real numbers having the β-expansion of the form $d_\beta(x) = x_kx_{k-1}\cdots x_1x_0 \bullet 0^\omega$.

$(-\beta)$-expansions, a numeration system built in analogy with Rényi β-expansions, was introduced in 2009 by Ito and Sadahiro (see [5]). They gave a lexicographic criterion for deciding whether some digit string is the $(-\beta)$-expansion of some x and also described several properties of $(-\beta)$-expansions concerning symbolic dynamics and ergodic theory. Note that dynamical properties of $(-\beta)$-expansions were also studied by Frougny and Lai (see [4]). We take the liberty of defining $(-\beta)$-expansions in a more general way, while an analogy with positive base numeration can still be easily seen.

Definition 1. *Let $-\beta < -1$ be a base and consider $x \in [l, l+1)$, where $l \in \mathbb{R}$ is arbitrary fixed. We define the $(-\beta)$-expansion of x as the digit string $d(x) = x_1x_2x_3\cdots$, with digits x_i given by*

$$x_i = \lfloor -\beta T^{i-1}(x) - l \rfloor , \tag{1}$$

where $T(x)$ stands for the generalised $(-\beta)$-transformation

$$T : [l, l+1) \to [l, l+1), \quad T(x) = -\beta x - \lfloor -\beta x - l \rfloor . \tag{2}$$

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It holds that

$$x = \frac{x_1}{-\beta} + \frac{x_2}{(-\beta)^2} + \frac{x_3}{(-\beta)^3} + \cdots$$

and the fractional point is again used in the notation, $d(x) = 0 \bullet x_1 x_2 x_3 \cdots$.

The set of digits used in $(-\beta)$-expansions of numbers (in the latter referred to as the alphabet of $(-\beta)$-expansions) depends on the choice of l and can be calculated directly from (1) as

$$\mathscr{A}_{-\beta,l} = \left\{ \lfloor -l(\beta+1) - \beta \rfloor, \dots, \lfloor -l(\beta+1) \rfloor \right\}. \quad (3)$$

We may demand that the numeration system possesses various properties. Let us summarise the most natural ones:

- The most common requirement is that zero is an allowed digit. We see that $0 \in \mathscr{A}_{-\beta,l}$ is equivalent to $0 \in [l, l+1)$ and consequently $l \in (-1, 0]$. Note that this implies $d(0) = 0 \bullet 0^\omega$.
- We may require that $\mathscr{A}_{-\beta,l} = \{0, 1, \dots, \lfloor \beta \rfloor\}$. This is equivalent to the choice $l \in \left(-\frac{\lfloor\beta\rfloor+1}{\beta+1}, -\frac{\beta}{\beta+1}\right]$.
- So far, $(-\beta)$-expansions were defined only for numbers from $[l, l+1)$. In Rényi numeration, the β-expansion of arbitrary $x \in \mathbb{R}^+$ (expansions of negative numbers differ only by "$-$" sign) is defined as $d_\beta(x) = x_k x_{k-1} \cdots x_1 x_0 \bullet x_{-1} x_{-2} \cdots$, where $k \in \mathbb{N}$ satisfies $\frac{x}{\beta^k} \in [l, l+1)$ and $d_\beta\left(\frac{x}{\beta^k}\right) = 0 \bullet x_k x_{k-1} x_{k-2} \cdots$. The same procedure does not work for $(-\beta)$-expansions in general. A necessary and sufficient condition for the existence of unique $d(x)$ for all $x \in \mathbb{R}$ is that $-\frac{1}{\beta}[l, l+1) \subset [l, l+1)$. This is equivalent to the choice $l \in \left(-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}\right]$. Note that this choice is disjoint with the previous one, so one cannot have uniqueness of $(-\beta)$-expansions and non-negative digits bounded by β at the same time.

Let us stress that in the following we will need 0 to be a valid digit. Therefore, we shall always assume $l \in (-1, 0]$. Note that we may easily derive that the digits in the alphabet $\mathscr{A}_{-\beta,l}$ are then bounded by $\lceil \beta \rceil$ in modulus.

3 Admissibility

In Rényi numeration there is a natural correspondence between ordering on real numbers and lexicographic ordering on their β-expansions. In $(-\beta)$-expansions, standard lexicographic ordering is not suitable anymore, hence a different ordering on digit strings is needed.

The so-called alternate order was used in the admissibility condition by Ito and Sadahiro and it will work also in the general case. Let us recall the definition. For the strings

$$u, v \in (\mathscr{A}_{-\beta,l})^{\mathbb{N}}, \quad u = u_1 u_2 u_3 \cdots \quad \text{and} \quad v = v_1 v_2 v_3 \cdots$$

we say that $u \prec_{alt} v$ (u is less than v in the alternate order) if $u_m(-1)^m < v_m(-1)^m$, where $m = \min\{k \in \mathbb{N} \mid u_k \neq v_k\}$. Note that standard ordering between reals in $[l, l+1)$ corresponds to the alternate order on their respective $(-\beta)$-expansions.

Definition 2. *An infinite string $x_1 x_2 x_3 \cdots$ of integers is called $(-\beta)$-admissible (or just admissible), if there exists an $x \in [l, l+1)$ such that $x_1 x_2 x_3 \cdots$ is its $(-\beta)$-expansion, i.e. $x_1 x_2 x_3 \cdots = d(x)$.*

We give the criterion for $(-\beta)$-admissibility (proven in [2]) in a form similar to both Parry lexicographic condition (see [6]) and Ito-Sadahiro admissibility criterion (see [5]).

Theorem 3. ([2]) *An infinite string* $x_1x_2x_3\cdots$ *of integers is* $(-\beta)$*-admissible, if and only if*

$$l_1l_2l_3\cdots \preceq_{alt} x_ix_{i+1}x_{i+2}\cdots \prec_{alt} r_1r_2r_3\cdots\,, \qquad \text{for all } \ i\geq 1, \tag{4}$$

where $l_1l_2l_3\cdots = d(l)$ *and* $r_1r_2r_3\cdots = d^*(l+1) = \lim_{\varepsilon\to 0+} d(l+1-\varepsilon)$.

Remark 4. *Ito and Sadahiro have described the admissibility condition for their numeration system considered with* $l = -\frac{\beta}{\beta+1}$. *This choice imply for any* β *the alphabet of the form* $\mathscr{A}_{-\beta,l} = \{0,1,\ldots,\lfloor\beta\rfloor\}$. *They have shown that in this case the reference strings used in the condition in Theorem 3 (i.e.* $d(l) = l_1l_2l_3\cdots$ *and* $d^*(l+1) = r_1r_2r_3\cdots$*) are related in the following way:*

$$r_1r_2r_3\cdots = 0l_1l_2l_3\cdots$$

if $d(l)$ *is not purely periodic with odd period length, and,*

$$r_1r_2r_3\cdots = \big(0l_1l_2\cdots l_{q-1}(l_q-1)\big)^{\omega}\,,$$

if $d(l) = \big(l_1l_2\cdots l_q\big)^{\omega}$, *where q is odd.*

Remark 5. *Besides Ito-Sadahiro case and the general one, we may consider another interesting example, the choice* $l = -\frac{1}{2}$, $\beta \notin 2\mathbb{Z}+1$. *This leads to a numeration defined on "almost symmetric" interval* $[-\frac{1}{2},\frac{1}{2})$ *with symmetric alphabet*

$$\mathscr{A}_{-\beta,-\frac{1}{2}} = \Big\{\overline{\Big\lfloor\frac{\beta+1}{2}\Big\rfloor},\ldots,\overline{1},0,1,\ldots\Big\lfloor\frac{\beta+1}{2}\Big\rfloor\Big\}\,.$$

Note that we use the notation $(-a) = \overline{a}$ *for shorter writing of negative digits. If we denote the reference strings as usual, i.e.* $d\big(-\frac{1}{2}\big) = l_1l_2l_3\cdots$ *and* $d^*\big(\frac{1}{2}\big) = r_1r_2r_3\cdots$, *the following relation can be shown:*

$$r_1r_2r_3\cdots = \overline{l_1l_2l_3\cdots}$$

if $d(l)$ *is not purely periodic with odd period length, and,*

$$r_1r_2r_3\cdots = \big(\overline{l_1l_2\cdots l_{q-1}(l_q-1)}l_1l_2\cdots l_{q-1}(l_q-1)\big)^{\omega}\,,$$

if $d(l) = \big(l_1l_2\cdots l_q\big)^{\omega}$, *where q is odd.*

4 $(-\beta)$-integers

We have already discussed basic properties of $(-\beta)$-expansions and the question of admissibility of digit strings. In the following, $(-\beta)$-admissibility will be used to define the set of $(-\beta)$-integers.

Let us define a "value function" γ. Consider a finite digit string $x_{k-1}\cdots x_1x_0$, then $\gamma(x_{k-1},\cdots x_1x_0) = \sum_{i=0}^{k-1} x_i(-\beta)^i$.

Definition 6. *We call* $x\in\mathbb{R}$ *a* $(-\beta)$*-integer, if there exists a* $(-\beta)$*-admissible digit string* $x_kx_{k-1}\cdots x_0 0^{\omega}$ *such that* $d(x) = x_kx_{k-1}\cdots x_1x_0 \bullet 0^{\omega}$. *The set of* $(-\beta)$*-integers is then defined as*

$$\mathbb{Z}_{-\beta} = \{x\in\mathbb{R} \mid x = \gamma(a_{k-1}a_{k-2}\cdots a_1a_0),\ a_{k-1}a_{k-2}\cdots a_1a_0 0^{\omega} \text{ is } (-\beta)\text{-admissible}\,,$$

or equivalently

$$\mathbb{Z}_{-\beta} = \bigcup_{i\geq 0}(-\beta)^i T^{-i}(0)\,.$$

Note that $(-\beta)$-expansions of real numbers are not necessarily unique. As was said before, uniqueness holds if and only if $l \in \left(-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}\right]$. Let us demonstrate this ambiguity on the following example.

Example 7. *Let β be the greater root of the polynomial x^2-2x-1, i.e. $\beta = 1+\sqrt{2}$, and let $[l,l+1) = \left[-\frac{\beta^9}{\beta^9+1}, \frac{1}{\beta^9+1}\right)$. Note that $[l,l+1)$ is not invariant under division by $(-\beta)$.*

If we want to find the $(-\beta)$-expansion of number $x \notin [l,l+1)$, we have to find such $k \in \mathbb{N}$ that $\frac{x}{(-\beta)^k} \in [l,l+1)$, compute $d\left(\frac{x}{(-\beta)^k}\right)$ by definition and then shift the fractional point by k positions to the right. The problem is that, in general, different choices of the exponent k may give different $(-\beta)$-admissible digit strings which all represent the same number x.

Let us find possible $(-\beta)$-expansions of 1. It can be shown that $\frac{1}{(-\beta)^k} \in [l,l+1)$ if and only if $k \in \mathbb{N}\setminus\{0,2,4,6,8\}$ and there are 5 $(-\beta)$-admissible digit strings representing 1, computed from $(-\beta)$-expansions of $\frac{1}{(-\beta)^k}$ for $k = 1,3,5,7,9$ respectively:

$$1\bullet 0^\omega \quad = \quad 120\bullet 0^\omega \quad = \quad 13210\bullet 0^\omega \quad = \quad 1322210\bullet 0^\omega \quad = \quad 132222210\bullet 0^\omega\,.$$

Let us mention some straightforward observations on the properties of $\mathbb{Z}_{-\beta}$:

- $\mathbb{Z}_{-\beta}$ is nonempty if and only if $0 \in \mathscr{A}_{-\beta,l}$, i.e. if and only if $l \in (-1,0]$.
- The definition implies $-\beta\mathbb{Z}_{-\beta} \subset \mathbb{Z}_{-\beta}$.
- A phenomenon unseen in Rényi numeration arises, there are cases when the set of $(-\beta)$-integers is trivial, i.e. when $\mathbb{Z}_{-\beta} = \{0\}$. This happens if and only if both numbers $\frac{1}{\beta}$ and $-\frac{1}{\beta}$ are outside of the interval $[l,l+1)$. This can be reformulated as

$$\mathbb{Z}_{-\beta} = \{0\} \quad \Leftrightarrow \quad \beta < -\frac{1}{l} \text{ and } \beta \leq \frac{1}{l+1}\,,$$

  and it can be seen that the strictest limitation for β arises when $l = -\frac{1}{2}$. This implies for any choice of $l \in \mathbb{R}$:

$$\mathbb{Z}_{-\beta} \neq \emptyset \text{ and } \beta \geq 2 \quad \Rightarrow \quad \mathbb{Z}_{-\beta} \supsetneq \{0\}\,.$$

- It holds that $\mathbb{Z}_{-\beta} = \mathbb{Z}$ if and only if $\beta \in \mathbb{N}$.

Remark 8. *As was shown in Example 7, in a completely general case of $(-\beta)$-expansions, there is a problem with ambiguity. Because of this, in the following we shall limit ourselves to the choice $l \in \left[-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}\right]$. Note that we allow Ito-Sadahiro case $l = -\frac{\beta}{\beta+1}$, which also contains ambiguities, but only in countably many cases, which can be avoided by introducing a notion of strong $(-\beta)$-admissibility.*

Definition 9. *Let $x_1x_2x_3\cdots \in \mathscr{A}_{-\beta,l}$. We say that*

$$x_1x_2x_3\cdots \text{ is strongly } (-\beta)\text{-admissible} \quad \text{if} \quad 0x_1x_2x_3\cdots \text{ is } (-\beta)\text{-admissible}\,.$$

Remark 10. *Note that if $l \in \left(-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}\right]$, the notions of strong admissibility and admissibility coincide. In the case $l = -\frac{\beta}{\beta+1}$, the only numbers with non-unique expansions are those of the form $(-\beta)^k l$, which have exactly two possible expansions using digit strings $l_1l_2l_3\cdots$ and $1l_1l_2l_3\cdots$. While both are $(-\beta)$-admissible, only the latter is also strongly $(-\beta)$-admissible.*

In order to describe distances between adjacent $(-\beta)$-integers, we will study ordering of finite digit strings in the alternate order. Denote by $\mathscr{S}(k)$ the set of infinite $(-\beta)$-admissible digit strings such that erasing a prefix of length k yields 0^ω, i.e. for $k \geq 0$, we have

$$\mathscr{S}(k) = \{a_{k-1}a_{k-2}\cdots a_0 0^\omega \mid a_{k-1}a_{k-2}\cdots a_0 0^\omega \text{ is } (-\beta)\text{-admissible}\},$$

in particular $\mathscr{S}(0) = \{0^\omega\}$. For a fixed k, the set $\mathscr{S}(k)$ is finite. Denote by $\mathrm{Max}(k)$ the string $a_{k-1}a_{k-2}\cdots a_0 0^\omega$ which is maximal in $\mathscr{S}(k)$ with respect to the alternate order and by $\max(k)$ its prefix of length k, i.e. $\mathrm{Max}(k) = \max(k)0^\omega$. Similarly, we define $\mathrm{Min}(k)$ and $\min(k)$. Thus,

$$\mathrm{Min}(k) \preceq_{alt} r \preceq_{alt} \mathrm{Max}(k), \qquad \text{for all digit strings } r \in \mathscr{S}(k).$$

With this notation we can give a theorem describing distances in $\mathbb{Z}_{-\beta}$ valid for cases $l \in \left[-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}\right]$. Note that for case $l = -\frac{\beta}{\beta+1}$ it was proven in [1].

Theorem 11. *Let $x < y$ be two consecutive $(-\beta)$-integers. Then there exist a finite string w over the alphabet $\mathscr{A}_{-\beta,l}$, a non-negative integer $k \in \{0,1,2,\dots\}$ and a positive digit $d \in \mathscr{A}_{-\beta,l} \setminus \{0\}$ such that $w(d-1)\mathrm{Max}(k)$ and $wd\mathrm{Min}(k)$ are strongly $(-\beta)$-admissible strings and*

$$\begin{array}{llll} x = \gamma(w(d-1)\max(k)) & < & y = \gamma(wd\min(k)) & \text{for } k \text{ even},\\ x = \gamma(wd\min(k)) & < & y = \gamma(w(d-1)\max(k)) & \text{for } k \text{ odd}. \end{array}$$

In particular, the distance $y - x$ between these $(-\beta)$-integers depends only on k and equals to

$$\Delta_k := \Big|(-\beta)^k + \gamma\big(\min(k)\big) - \gamma\big(\max(k)\big)\Big|. \tag{5}$$

5 Coding $\mathbb{Z}_{-\beta}$ by an infinite word

Note that in order to get an explicit formula for distances from Theorem 3, knowledge of reference strings $\min(k)$ and $\max(k)$ is necessary. These depend on both reference strings $d(l)$ and $d^*(l+1)$. Concerning the form of $\min(k)$ and $\max(k)$ we provide the following proposition.

Proposition 12. *Let $\beta > 1$. Denote $d(l) = l_1 l_2 l_3 \cdots$, $d^*(l+1) = r_1 r_2 r_3 \cdots$.*

- $\min(0) = \max(0) = \varepsilon$,
- *for $k \geq 1$ either $\min(k) = l_1 l_2 \cdots l_k$ or there exists $m(k) \in \{0,\cdots,k-1\}$ such that*

$$\min(k) = \begin{cases} l_1 l_2 \cdots (l_{k-m(k)}+1)\min(m(k)) & \text{if } k-m(k) \text{ even}\\ l_1 l_2 \cdots (l_{k-m(k)}-1)\max(m(k)) & \text{if } k-m(k) \text{ odd}\end{cases}$$

- *for $k \geq 1$ either $\max(k) = r_1 r_2 \cdots r_k$ or there exists $m'(k) \in \{0,\cdots,k-1\}$ such that*

$$\max(k) = \begin{cases} r_1 r_2 \cdots (r_{k-m'(k)}-1)\max(m'(k)) & \text{if } k-m'(k) \text{ even}\\ r_1 r_2 \cdots (r_{k-m'(k)}+1)\min(m'(k)) & \text{if } k-m'(k) \text{ odd}\end{cases}$$

Computing $\min(k)$ and $\max(k)$ for a general choice of l may lead to difficult discussion, however, in special cases an important relation between $d(l)$ and $d^*(l+1)$ arises and eases the computation. Examples were given in Remarks 4 and 5.

Let us now describe how we can code the set of $(-\beta)$-integers by an infinite word over the infinite alphabet $\mathbb{N}$.

Let $(z_n)_{n\in\mathbb{Z}}$ be a strictly increasing sequence satisfying

$$z_0 = 0 \quad \text{and} \quad \mathbb{Z}_{-\beta} = \{z_n \mid n \in \mathbb{Z}\}.$$

We define a bidirectional infinite word over an infinite alphabet $\mathbf{v}_{-\beta} \in \mathbb{N}^{\mathbb{Z}}$, which codes the set of $(-\beta)$-integers. According to Theorem 11, for any $n \in \mathbb{Z}$ there exist a unique $k \in \mathbb{N}$, a word w with prefix 0 and a letter d such that

$$z_{n+1} - z_n = \big|\gamma(w(d-1)\max(k)) - \gamma(wd\min(k))\big|.$$

We define the word $\mathbf{v}_{-\beta} = (v_i)_{i\in\mathbb{Z}}$ by $v_n = k$.

Theorem 13. *Let $\mathbf{v}_{-\beta}$ be the word associated with $(-\beta)$-integers. There exists an antimorphism $\Phi : \mathbb{N}^* \to \mathbb{N}^*$ such that $\Psi = \Phi^2$ is a non-erasing non-identical morphism and $\Psi(\mathbf{v}_{-\beta}) = \mathbf{v}_{-\beta}$. Φ is always of the form*

$$\Phi(2l) = S_{2l}(2l+1)\widetilde{R_{2l}} \quad \textit{and} \quad \Phi(2l+1) = R_{2l+1}(2l+2)\widetilde{S_{2l+1}},$$

where $\widetilde{u}$ denotes the reversal of the word u and words R_j, S_j depend only on j and on $\min(k), \max(k)$ with $k \in \{j, j+1\}$.

The proof is based on the self-similarity of $\mathbb{Z}_{-\beta}$, i.e. $-\beta\mathbb{Z}_{-\beta} \subset \mathbb{Z}_{-\beta}$, and on the following idea. Let $x = \gamma(w(d-1)\max(k)) < y = \gamma(wd\min(k))$ be two neighbours in $\mathbb{Z}_{-\beta}$ with gap Δ_k and suppose only k even. If we multiply both x and y by $(-\beta)$, we get a longer gap with possibly more $(-\beta)$-integers in between. It can be shown that between $-\beta y$ and $-\beta x$ there is always a gap Δ_{k+1}. Hence the description is of the form $\Phi(k) = S_k(k+1)\widetilde{R_k}$, where the word S_k codes the distances between $(-\beta)$-integers in $[\gamma(wd\min(k)0), \gamma(wd\min(k+1))]$ and, similarly, R_k encodes distances within the interval $[\gamma(w(d-1)\max(k)0), \gamma(w(d-1)\max(k+1))]$.

As it turns out, in some cases (mostly when reference strings $l_1l_2l_3\cdots$ and $r_1r_2r_3\cdots$ are eventually periodic of a particular form) we can find a letter-to-letter projection to a finite alphabet $\Pi : \mathbb{N} \to \mathscr{B}$ with $\mathscr{B} \subset \mathbb{N}$, such that $\mathbf{u}_{-\beta} = \Pi\mathbf{v}_{-\beta}$ also encodes $\mathbb{Z}_{-\beta}$ and it is a fixed point of a an antimorphism $\varphi = \Pi \circ \Phi$ over the finite alphabet $\mathscr{B}$. Clearly, the square of φ is then a non-erasing morphism over $\mathscr{B}$ which fixes $\mathbf{u}_{-\beta}$.

Let us mention that $(-\beta)$-integers in the Ito-Sadahiro case $l = -\frac{\beta}{\beta+1}$ are also subject of [8]. For β with eventually periodic $d(l)$, Steiner finds a coding of $\mathbb{Z}_{-\beta}$ by a finite alphabet and shows, using only the properties of the $(-\beta)$-transformation, that the word is a fixed point of a non-trivial morphism. Our approach is of a combinatorial nature, follows a similar idea as in [1] and shows existence of an antimorphism for any base β.

To illustrate the results, let us conclude this contribution by an example.

Example 14. *Let β be the real root of $x^3 - 3x^2 - 4x - 2$ (β Pisot, ≈ 4.3) and $l = -\frac{1}{2}$. The admissibility condition gives us for any admissible digit string $(x_i)_{i\geq 0}$:*

$$201^\omega \preceq_{alt} x_ix_{i+1}x_{i+2}\cdots \prec_{alt} \overline{2}0\overline{1}^\omega \quad \textit{for all } x \geq 0.$$

We obtain

$$\min(0) = \varepsilon, \quad \min(1) = 2, \quad \min(2) = 20$$

and

$$\min(2k+1) = 20(11)^{k-1}0, \quad \min(2k+2) = 20(11)^k \quad \textit{for } k \geq 1\,.$$

Clearly it holds that $\max(i) = \overline{\min(i)}$ *for all* $i \in \mathbb{N}$.

Theorem 11 gives us the following distances within $\mathbb{Z}_{-\beta}$*:*

$$\Delta_0 = 1, \quad \Delta_1 = -1 + \frac{4}{\beta} + \frac{2}{\beta^2}, \quad \textit{and} \quad \Delta_{2k} = 1 - \frac{2}{\beta} - \frac{2}{\beta^2}, \quad \Delta_{2k+1} = 1 + \frac{2}{\beta} + \frac{2}{\beta^2} \quad \textit{for } k \geq 1\,.$$

Finally, the antimorphism $\Phi : \mathbb{N}^* \to \mathbb{N}^*$ *is given by*

$$\begin{aligned} 0 &\to 0^2 10^2\,, \\ 1 &\to 2\,, \\ 2 &\to 3\,, \end{aligned}$$

and for $k \geq 1$

$$\begin{aligned} 2k+1 &\to 0^2 10(2k+2)010^2\,, \\ 2k+2 &\to 2k+3\,. \end{aligned}$$

It can be easily seen that a projection from $\mathbb{N}$ *to a finite alphabet exists and a final antimorphism* φ : $\{0,1,2,3\}^* \to \{0,1,2,3\}^*$ *is of the form*

$$\begin{aligned} 0 &\to 0^2 10^2, \\ 1 &\to 2, \\ 2 &\to 3, \\ 3 &\to 0^2 102010^2. \end{aligned}$$

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