SCHWARZ ANALYSIS OF AN ITERATIVE SUBSTRUCTURING ALGORITHM FOR P1 NONCONFORMING FINITE ELEMENTS

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Abstract

We give an iterative substructuring algorithm for the P1 nonconforming finite element approximation of a second order elliptic problem in dimension two. The algorithm and its convergence analysis are based on an abstract framework of the Schwarz method. A bound is established for the condition number of this iterative method which grows only logarithmically with the number of degrees of freedom. We olso give an application of this method to the lowest order Raviart-Thomas mixed method.

Key words. elliptic boundary value problem, domain decomposition method, substructuring algorithm, Schwarz method, nonconforming finite element, mixed finite element

AMS(MOS) subject classifications. 65N30, 65F10

1 Introduction

Domain decomposition algorithms are preconditioned iterative methods where the preconditioners are constructed from solvers for the given partial differential equation restricted to subregions into which the given region is subdivided. The iterative substructuring methods, based on decompositions of the region into nonoverlapping subregions form one of the main families of such algorithms.

Many domain decomposition algorithms can be described and analyzed as Schwarz methods, which are generalizations of the alternating method of Schwarz [22] discovered more then 120 years ago. Schwarz methods have primarily been associated with a division of the region into overlapping subregions. Recently, it has been known that the iterative substructuring methods also fit well into a common Schwarz framework, see Dryja and Widlund [11]. It will be the point of view adopted in this paper.

The purpose of this paper is to develop a domain decomposition method for second order elliptic partial differential equations approximated by a P1 nonconforming finite element method.

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There is some recent work in this area when the overlap between subregions is generous. Sarkis [21] has developed an additive Schwarz method which is insensitive to the jumps in coefficients but converges in a quasi optimal rate. Cowsar [8] and Brenner [4] have obtained an optimal rate of convergence using different approaches.

An important difference between the nonconforming and the conforming finite element method is that in the first case there are no nodes at the vertices of the subregions. To overcome this difficulty we follow closely the approach of Brenner [4] which is based on the connection between the nonconforming finite element space and an appropriate conforming finite element space.

The remainder of this paper is organized as follows. We summarize some abstract results on the convergence of the multiplicative and additive Schwarz methods in §2. In §3 we apply these algorithms to a model problem and derive the convergence estimates. In §3 we show how these methods can be used for the Raviart-Thomas mixed finite element method by taking advantage on a nonconforming equivalent formulation given in [1]. See [13, 12, 16, 17, 18] for related works.

Through out this paper, we will use the following notations. Let Δ denote any open interval of the real line or any polygonal domain in R^2. For any real number s, we consider the classical hilbertian Sobolev space $H^s(\Delta)$, provided with the usual norm $\|\cdot\|_{s,\Delta}$. For any real number $s \geq 0$, $H_0^s(\Delta)$ stands for the closure in $H^s(\Delta)$ of the space of indefinitely differentiable functions with a compact support in Δ. Finally, we recall that the interpolation space with index 1/2 between $H_0^1(\Delta)$ and $L^2(\Delta)$ is $H_{00}^{1/2}(\Delta)$ (see Lions, Magenes [15]); we denote its norm $\|\cdot\|_{H_{00}^{1/2}(\Delta)}$.

2 Abstract theory for Schwarz methods

Let $a(\cdot,\cdot)$ be a symmetric and positive definite bilinear form on a Hilbert space V. Assume that $V_i, i = 0, \ldots, J$, are closed subspaces of V satisfying

$$V = \sum_{i=0}^{J} V_i. \tag{1}$$

Let P_i be the orthogonal projection operator on V_i with respect to the form $a(\cdot,\cdot)$; i.e.,

$$a(P_i u - u, v) = 0 \quad \forall v \in V_i.$$

For L, a fixed element in the dual space of V, the Schwarz alternating method to solve the problem

$$\begin{cases} \text{find } u \in V; \\ a(u,v) = L(v) \quad \forall v \in V, \end{cases} \tag{2}$$

is described below.

Given $u^n \in V$, an approximation of u, we compute the next approximation $u^{n+1} \in V$ in three steps:

1. Set $Y_{-1} = u^n$.

2. Define $Y_i \in V$ for $i = 0, \ldots, J$ by

$$Y_i = Y_{i-1} + P_i(u - Y_{i-1}).$$

3. Set $u^{n+1} = Y_J$.

Note that for $i = 0, \ldots, J$, the term $v_i = P_i(u - Y_{i-1})$ is calculated by solving the following subproblem

$$\begin{cases} \text{find } v_i \in V_i; \\ a(v_i, \varphi) = L(\varphi) - a(Y_{i-1}, \varphi) \quad \forall \varphi \in V_i. \end{cases} \tag{3}$$

It is easy to show that the error propagation operator is

$$E_J = (id - P_J)(id - P_{J-1}) \cdots (id - P_0). \tag{4}$$

Estimates for the rate of convergence of the iterative method follow directly from bounds on the norm of E_J. These bounds will depend on the following parameters.

Let $\mathcal{E} = \{\varepsilon_{ij}\}_{1 \le i,j \le J}$ be the matrix of strengthened Cauchy-Schwarz coefficients defined by

$$|a(v_i, v_j)| \le \varepsilon_{ij} a(v_i, v_i)^{1/2} a(v_j, v_j)^{1/2} \quad \forall v_i \in V_i,\ \forall v_j \in V_j\ i, j = 1, \cdots, J.$$

Let C_0 be the minimum constant such that for all $v \in V$ there exists a representation, $v = \sum_{i=0}^{J} v_i$, $v_i \in V_i$, with

$$\sum_{i=0}^{J} a(v_i, v_i) \le C_0^2 a(v, v) \quad \forall v \in V.$$

We have the following convergence result for the multiplicative algorithm.

Theorem 2.1 *Let $\rho(\mathcal{E})$ be the spectral radius of the matrix $\mathcal{E}$. The rate of convergence of the multiplicative Schwarz algorithm is bounded by*

$$\|E_J\| \le \sqrt{1 - \frac{1}{(2\rho(\mathcal{E})^2 + 1)C_0^2}}.$$

The additive version of the Schwarz alternative method is based upon the classical Schwarz method, but modified so that the resulting algorithm is easier to parallelize. Define the linear operator P by

$$P = \sum_{i=0}^{J} P_i;$$

note that since P is the sum of $a(\cdot,\cdot)$-orthogonal projections, P is symmetric in the inner product $a(\cdot,\cdot)$. Furthermore, P is positive definite, see Theorem 2.2 below. Then one can use the conjugate gradient method to solve the equation

$$Pu = g_a.$$

In order to retain the same solution u of problem (2), we must find an appropriate right hand side g_a. Fortunately, it is computable:

$$g_a = \sum_{i=0}^{J} g_i,$$

where g_i is the solution of the following subproblem:

$$\begin{cases} \text{find } g_i \in V_i; \\ a(g_i, \varphi) = L(\varphi) \quad \forall \varphi \in V_i. \end{cases} \tag{5}$$

When analyzing the conjugate gradient algorithm, the crucial issue is the spectral condition number $\kappa(P)$ of the iteration operator. It is well known that $\kappa = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$, where

$$\lambda_{\min}(P) \leq \frac{a(Pu, u)}{a(u, u)} \leq \lambda_{\max}(P)$$

are the maximum and minimum eigenvalues of P given by the extrema of the Rayleigh quotient of P.

Theorem 2.2 *For all $u \in V$*

$$(1/C_0^2) a(u, u) \leq a(Pu, u) \leq (\rho(\mathcal{E}) + 1) a(u, u).$$

Thus $\kappa(P) \leq (\rho(\mathcal{E}) + 1) C_0^2$.

Note that we can also regard the multiplicative algorithm as an iterative method for solving the equation

$$T_m u = g_m,$$

for an adequate right hand side g_m and where the operator T_m satisfies

$$T_m = id - E_J.$$

This is a nonsymmetric operator, the equation can be solved with GMRES.

Results for the additive form are due primarily to Dryja and Widlund [11] and Nepomnyaschikh [19], while the result for multiplicative Schwarz methods is a variant of results of Bramble, Pasciak, Wang, and Xu [3], and Xu [25], see also Lions [14] for early work on the case of two subspaces.

3 Scalar P1 nonconforming finite element

3.1 The elliptic problem and its discretization

Throughout this paper, we use boldface capital letters to denote matrix-valued functions of two independent variables and boldface small letters to denote vector-valued functions.

Let Ω be a polygonal domain in R^2, $f \in L^2(\Omega)$, and $\mathbf{A}$ a sufficiently smooth symmetric matrix-valued function on $\overline{\Omega}$ such that for a positive constant α

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{2} \mathbf{A}_{ij}(x)\xi_i\xi_j \qquad \forall x \in \overline{\Omega},\ \forall \xi \in R^2.$$

Consider the second order partial differential equation

$$\begin{cases} -\operatorname{div} \mathbf{A}(\mathbf{grad} u) & = f \quad \text{in } \Omega, \\ \qquad\qquad u & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{6}$$

The variational formulation for (6) is to find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \int_\Omega fv\,dx \quad \forall v \in H_0^1(\Omega) \tag{7}$$

where

$$a(u,v) = \int_\Omega \mathbf{A}\mathbf{grad} u \cdot \mathbf{grad} v\,dx.$$

A coarse triangulation τ_H is introduced by dividing Ω into nonoverlapping triangles Ω_i, $i = 1, \cdots, J$ also called substructures. The substructures Ω_i are further divided into elements in such a way that a conforming triangulation τ_h of all Ω is obtained. We associate parameters H and h with these coarse and fine triangulations and assume that all the substructures and elements are shape regular in the usual sense. Throughout this paper C (with or without subscripts) will denote a generic positive constant independent of the mesh parameters h and H.

For each nonnegative integer k, denote by $P^k(T)$ the space of polynomial functions of degree $\leq k$ on T. We introduce the following spaces

$$M_{-1}^k(\tau_h) = \{v \in L^2(\Omega); \quad v|_{\overline{T}} \in P^k(T) \quad \forall T \in \tau_h\}.$$

and

$$M_0^k(\tau_h) = M_{-1}^k(\tau_h) \cap H_0^1(\Omega).$$

We denote by $\mathcal{E}_h$ the set of edges of the triangles in τ_h and set $\mathcal{E}_h^\partial = \{e \in \mathcal{E}_h; \quad e \subset \partial\Omega\}$ and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \mathcal{E}_h^\partial$.

We define

$$V^h = \{v \in M_{-1}^1(\tau_h); \quad v \text{ is continuous at the midpoints of each } e \in \mathcal{E}_h^0 \text{ and vanishes at the midpoints of each } e \in \mathcal{E}_h^\partial\},$$

equipped with the norm

$$\|v\|_{V^h}^2 = \sum_T \int_T |\mathbf{grad} v|^2\,dx.$$

The P1 nonconforming finite element approximation of (7) is to find $u_h \in V^h$ such that

$$a_h(u_h, v) = \int_\Omega fv\,dx \quad \forall v \in V^h, \tag{8}$$

where $a_h(\cdot,\cdot)$ is the continuous bilinear form on $V^h + H_0^1(\Omega)$ defined by

$$a_h(u,v) = \sum_T \int_T \mathbf{Agrad}u \cdot \mathbf{grad}v \, dx.$$

This bilinear form is obviously symmetric and uniformly positive definite on V^h. Then, the discrete problem is well posed. Convergence results and error estimates can be found in [9].

3.2 Convergence analysis

We introduce the following notation

$$\Sigma^H = \{x; \text{ nodes of } \tau^H\},$$

and

$$\Lambda^E = \{(i,j);\, x_i, x_j \in \Sigma^H;\;\; x_i, x_j \text{ adjacent and not both on } \partial\Omega\}.$$

We also denote by Γ_{ij} the edge common to two adjoining rectangles Ω_i and Ω_j and let $\Omega_{ij} = \Omega_i \cup \Gamma_{ij} \cup \Omega_j$. The set $\{\Omega_{ij},\, (i,j) \in \Lambda^E\}$ forms a decomposition of the region Ω into overlapping subregions. We associate to each Ω_{ij} the subspace

$$V_{ij}^h = \{\varphi \in V^h; \quad \text{supp}\varphi \subset \overline{\Omega}_{ij}\}.$$

We will show that

$$V^h = M_0^1(\tau_H) + \sum_{(i,j)\in\Lambda^E} V_{ij}^h,$$

where $M_0^1(\tau_H)$ is defined similarly with respect to τ_H. As described in section 2, we associate to this decomposition either the multiplicative or the additive Schwarz method.

We follow Brenner [4] and introduce the operators

$$E_h : V^h \longrightarrow M_0^2(\tau_h)$$
$$F_h : M_0^2(\tau_h) \longrightarrow V_h$$

defined by

$$(E_h v)(m) = v(m) \text{ for all internal midpoints } m \in \tau_h$$
$$(E_h v)(p) = \text{average of } v_i(p) \text{ for all internal vertices } p \in \tau_h$$

where $v_i = v|_{T_i}$ and $T_i \in \tau_h$ contains p at a vertex,

$$(F_h w)(m) = w(m) \text{ for all midpoints in } \tau_h.$$

We will also need the nodal interpolation operator

$$I_H : M_0^2(\tau_h) \longrightarrow M_0^1(\tau_H)$$

given by

$$(I_H w)(p) = w(p) \text{ for all internal vertices } p \in \tau_H.$$

The following result concerning the stability of the intergrid transfer operators E_h and F_h can be found in [4].

Lemma 3.1 *The following estimates on E_h and F_h hold:*

$$|E_h v|_{1,\Omega} \le C\|v\|_{V^h} \quad \forall v \in V^h$$
$$\|F_h w\|_{V^h} \le C|w|_{1,\Omega} \quad \forall w \in M_0^2(\tau_h).$$

We will also need some technical results usually used in the theory of domain decomposition methods. The first one is an estimate on the interpolation operator I_H proved in [2, 24].

Lemma 3.2 *The following estimate holds:*

$$|I_H w|_{1,\Omega}^2 \le C(1+\log\frac{H}{h})|w|_{1,\Omega}^2 \quad \forall w \in M_0^2(\tau_h).$$

The following lemma is given in [2, 24] for continuous piecewise linear functions. The same result is valid for continuous piecewise quadratic functions with no modifications in the proof.

Lemma 3.3 *The following estimate holds:*

$$\|w - I_H w\|_{H_{00}^{1/2}(\Gamma_{ij})}^2 \le C(1+\log\frac{H}{h})^2|w|_{1,\Omega}^2 \quad \forall w \in M_0^2(\tau_h).$$

Finally, we give an estimate of the discrete $a_h(\cdot,\cdot)$-harmonic extension of boundary values obtained by replacing the value on $\partial\Omega_i$ by zero except on Γ_{ij}. More precisely, let

$$\Phi(\Gamma_{ij}) = \{v|_{\Gamma_{ij}};\ v \in M_0^2(\tau_h)\}.$$

Given a function $\varphi \in \Phi(\Gamma_{ij}) \cap H_0^1(\Gamma_{ij})$ we characterize its $a_h(\cdot,\cdot)$-harmonic extension $R^i_{\Gamma_{ij}}\varphi \in \{v|_{\Omega_i};\ v \in M_0^2(\tau_h)\}$ by

$$\begin{cases} a_h(R^i_{\Gamma_{ij}}\varphi, v) = 0 \quad \forall v \in M_0^2(\tau_h) \cap H_0^1(\Omega_i) \\ R^i_{\Gamma_{ij}}\varphi = \varphi \ on\ \Gamma_{ij} \\ R^i_{\Gamma_{ij}}\varphi = 0 \ on\ \partial\Omega_i \backslash \Gamma_{ij}. \end{cases} \tag{9}$$

The following result can be found in [2].

Lemma 3.4 *There exists a positive constant C such that*

$$a_h(R^i_{\Gamma_{ij}}\varphi, R^i_{\Gamma_{ij}}\varphi) \le C\|\varphi\|_{H_{00}^{1/2}(\Gamma_{ij})}.$$

We are now in position to prove the main result.

Theorem 3.1 *There exists a constant C such that for each $v \in V^h$ there exists a decomposition*

$$v = v_0 + \sum_{ij} v_{ij}$$

with $v_0 \in M_0^1(\tau_H)$ and $v_{ij} \in V_{ij}^h$ satisfying the inequality

$$a_h(v_0, v_0) + \sum_{ij} a_h(v_{ij}, v_{ij}) \le C(1+\log\frac{H}{h})^2 a_h(v,v).$$

Proof. Let $v \in V^h$, by definition $E_h v \in M_0^2(\tau_h)$. Denote $\omega = E_h v - I_H(E_h v)$. We will show that

$$\omega = \sum_{ij} \omega_{ij},$$

with $\omega_{ij} \in M_0^2(\tau_h) \cap H_0^1(\Omega_{ij})$. Then

$$E_h v = I_H(E_h v) + \sum_{ij} \omega_{ij}.$$

Noting that $F_h \circ E_h = id$ and $F_h \circ I_H = I_H$, we apply the operator F_h to the last equality and arrive to

$$v = I_H(E_h v) + \sum_{ij} F_h \omega_{ij}.$$

We take $v_0 = I_H(E_h v) \in M_0^1(\tau_H)$ and $v_{ij} = F_h \omega_{ij} \in V_{ij}^h$. By virtue of Lemma 3.2 and the equivalency of the norms $\|\cdot\|_{V^h}$ and $a_h(\cdot,\cdot)^{1/2}$

$$a_h(v_0, v_0) \leq C(1 + \log \frac{H}{h}) a_h(E_h v, E_h v).$$

Now using the estimates given in Lemma 3.1 for the operators E_h and F_h we immediately obtain

$$a_h(v_0, v_0) + \sum_{ij} a_h(v_{ij}, v_{ij}) \leq C((1 + \log \frac{H}{h}) a_h(v, v) + \sum_{ij} a_h(\omega_{ij}, \omega_{ij})).$$

To conclude, it suffices to show that the functions ω_{ij} can be constructed in such a way that the following estimate holds

$$\sum_{ij} a_h(\omega_{ij}, \omega_{ij}) \leq C(1 + \log \frac{H}{h})^2 a_h(v, v).$$

Let us construct ω_{ij}. For all $k = 1, \cdots, J$ we denote $\alpha_k = 1/(\text{number of interior sides of}\, \Omega_k)$ and P_k the $a_h(\cdot,\cdot)$ -orthogonal projection onto $M_0^2(\tau_h) \cap H_0^1(\Omega_k)$. We consider the projection operator P_{ij} defined by $P_{ij} = P_i \oplus P_j$. Finally, let $P_{ij}^{\perp}$ be the orthogonal of P_{ij} in $H_0^1(\Omega_{ij}) \cap M_0^2(\tau_h)$. We take

$$\omega_{ij} = \alpha_i P_i(\omega) + \alpha_j P_j(\omega) + P_{ij}^{\perp}(\omega).$$

It easy to show that $\omega = \sum_{ij} \omega_{ij}$. By the properties of the orthogonal projection

$$\begin{aligned} a_h(\omega_{ij}, \omega_{ij}) &= \alpha_i^2 a_h(P_i\omega, P_i\omega) + \alpha_j^2 a_h(P_j\omega, P_j\omega) + a_h(P_{ij}^{\perp}\omega, P_{ij}^{\perp}\omega) \\ &\leq \alpha_i^2 a_h^i(\omega, \omega) + \alpha_j^2 a_h^j(\omega, \omega) + a_h(P_{ij}^{\perp}\omega, P_{ij}^{\perp}\omega), \end{aligned}$$

where we have denoted $a_h^i(\cdot,\cdot)$ the restriction of $a_h(\cdot,\cdot)$ to Ω_i.

It is straightforward to see that $P_{ij}^{\perp}$ is equal to $R_{\Gamma_{ij}}^i(\omega)$ in Ω_i and $R_{\Gamma_{ij}}^j(\omega)$ in Ω_j. Applying Lemma 3.4 we obtain the inequality

$$a_h(\omega_{ij}, \omega_{ij}) \leq \alpha_i^2 a_h^i(\omega, \omega) + \alpha_j^2 a_h^j(\omega, \omega) + C\|w|_{\Gamma_{ij}}\|^2_{H_{00}^{1/2}(\Gamma_{ij})}.$$

By definition $w = (E_h v - I_H(E_h v))$; the triangular inequality and Lemmas 3.2 and 3.3 lead to

$$a_h(\omega_{ij}, \omega_{ij}) \le C(1 + \log \frac{H}{h})^2 (|E_h v|^2_{1,\Omega_i} + |E_h v|^2_{1,\Omega_j}).$$

Summing over $(i,j) \in \Lambda^E$ and using again the stability property of E_h given in Lemma 3.1 we arrive to

$$\sum_{ij} a_h(\omega_{ij}, \omega_{ij}) \le C(1 + \log \frac{H}{h})^2 \|v\|^2_{V^h},$$

and we conclude by the equivalency of the norms $\|\cdot\|_{V^h}$ and $a_h(\cdot,\cdot)^{1/2}$. $\Box$

To obtain an upper bound on $\rho(\mathcal{E})$, we observe that we can color $\{\Omega_{ij},\, (i,j) \in \Lambda^E\}$ in such a way that subdomains of the same color are disjoint. It is easy to see that no more than 3 colors are needed. In addition, we have the global space $M_0^1(\tau_H)$ that intersects all other subspaces and which we associate with a forth color. Noting that two elements belonging to (different) subspaces associated to subdomains of the same color are $a_h(\cdot,\cdot)$-orthogonal, it follows that $\rho(\mathcal{E}) < 4$.

4 The lowest order Raviart-Thomas mixed method

4.1 The mixed formulation of the model problem and its discretization

In this section we use the same notation introduced previously and the model problem (6) will be approximated by a mixed finite element method. The flux variable $\mathbf{p} = -\mathbf{A}(\mathbf{grad} u)$ is of interest in many physical problems. The Raviart-Thomas method is useful when an accurate approximation of this flux variable is required.

We consider the space

$$H(\mathrm{div}, \Omega) = \{\mathbf{q} \in (L^2(\Omega))^2; \mathrm{div}\, \mathbf{q} \in L^2(\Omega)\},$$

equipped with the usual graph norm,

$$\|\mathbf{q}\|^2_{H(\mathrm{div},\Omega)} = \|\mathbf{q}\|^2_{0,\Omega} + \|\,\mathrm{div}\, \mathbf{q}\|^2_{0,\Omega},$$

which makes it a Hilbert space.

The mixed variational formulation of (6) is

$$\left\{ \begin{array}{ll} \text{find } (\mathbf{p}, u) \in H(\mathrm{div}, \Omega) \times L^2(\Omega); & \\ \displaystyle\int_\Omega \mathbf{C}\mathbf{p} \cdot \mathbf{q}\, dx - \int_\Omega u \,\mathrm{div}\, \mathbf{q}\, dx = 0 & \forall \mathbf{q} \in H(\mathrm{div}, \Omega), \\ \displaystyle\int_\Omega \mathrm{div}\, \mathbf{p} v\, dx = \int_\Omega f v\, dx & \forall v \in L^2(\Omega), \end{array} \right. \tag{10}$$

where $\mathbf{C} = \mathbf{A}^{-1}$. It is well known that this problem has a unique solution, see [20] or [23].

We associate with each element $T \in \tau_h$ the Raviart-Thomas space [20, 23] defined by :

$$R(T) = \{(a + cx, b + cy)^t, a, b, c \in R\},$$

where $(x, y)^t$ is the space variable. Now we introduce the space

$$R_{-1}^h = \{\mathbf{q} \in (L^2(\Omega))^2; \quad \mathbf{q}|_T \in R(T) \quad \forall T \in \tau_h\},$$

and the finite dimensional subspace of $H(\mathrm{div}, \Omega)$

$$R_0^h = R_{-1}^h \cap H(\mathrm{div}, \Omega),$$

in which we will seek an approximation $\mathbf{p}_h$ of the flux $\mathbf{p}$.

The finite dimensional space corresponding to the scalar variable is the space $M_{-1}^0(\tau_h)$ of piecewise constant functions.

The lowest order Raviart-Thomas finite element approximation of (10) is given by

$$\left\{\begin{array}{ll} \text{find } (\mathbf{p}_h, u_h) \in R_0^h \times M_{-1}^0(\tau_h); & \\ \displaystyle\int_\Omega \mathbf{C}\mathbf{p}_h \cdot \mathbf{q}\, dx - \int_\Omega u_h \,\mathrm{div}\, \mathbf{q}\, dx = 0 & \forall \mathbf{q} \in R_0^h, \\ \displaystyle\int_\Omega \mathrm{div}\, \mathbf{p}_h v\, dx = \int_\Omega f v\, dx & \forall v \in M_{-1}^0(\tau_h). \end{array}\right. \tag{11}$$

The spaces R_0^h and $M_{-1}^0(\tau_h)$ are constructed so that problem (11) is well posed, cf. [23], and furthermore

$$\lim_{h\to 0} \|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega} = \lim_{h\to 0} \|u - u_h\|_{0,\Omega} = 0.$$

Additional convergence results including error estimates can be found in [20] and the references contained therein.

4.2 An equivalent nonconforming formulation

Note that, by definition, the normal component of the elements in R_0^h is continuous across the interelement boundaries $e \in \mathcal{E}_h^0$. One may eliminate this continuity requirement from the finite element space and enforce it through Lagrange multipliers defined on the interelement boundaries.

Let $M_{-1}^0(\mathcal{E}_h^0)$ be the space of all functions on $\mathcal{E}_h$ which are constant functions when restricted to each edge $e \in \mathcal{E}_h^0$ and vanish on $\mathcal{E}_h^\partial$. It is easy to see that if $\mathbf{q} \in R_{-1}^h$, then

$$\mathbf{q} \in R_0^h \Leftrightarrow \sum_{T\in\tau_h} \int_{\partial T} \mathbf{q} \cdot \boldsymbol{\nu}_T \mu\, de = 0 \quad \forall \mu \in M_{-1}^0(\mathcal{E}_h^0),$$

where $\boldsymbol{\nu}_T$ denotes the exterior normal to T. As noted in [1] the extended problem

$$\left\{\begin{array}{ll} \text{find } (\overline{\mathbf{p}}_h, \overline{u}_h, \lambda_h) \in R_{-1}^h \times M_{-1}^0(\tau_h) \times M_{-1}^0(\mathcal{E}_h^0); & \\ \displaystyle\int_\Omega \mathbf{C}\overline{\mathbf{p}}_h \cdot \mathbf{q}\, dx - \sum_{T\in\tau_h} (\int_T \overline{u}_h \,\mathrm{div}\, \mathbf{q}\, dx - \int_{\partial T} \lambda_h \mathbf{q} \cdot \boldsymbol{\nu}_T\, de) = 0 & \forall \mathbf{q} \in R_{-1}^h, \\ \displaystyle\sum_{T\in\tau_h} \int_T \mathrm{div}\, \overline{\mathbf{p}}_h v\, dx = \int_\Omega f v\, dx & \forall v \in M_{-1}^0(\tau_h), \\ \displaystyle\sum_{T\in\tau_h} \int_{\partial T} \overline{\mathbf{p}}_h \cdot \boldsymbol{\nu}_T \mu\, de = 0 & \forall \mu \in M_{-1}^0(\mathcal{E}_h^0). \end{array}\right. \tag{12}$$

has a unique solution $(\overline{\mathbf{p}}_h, \overline{u}_h, \lambda_h)$. Moreover $\overline{\mathbf{p}}_h$ can be identified with $\mathbf{p}_h$, and $\overline{u}_h = u_h$.

The extended problem (12) can be written as a nonconforming finite element problem in the following manner. We first introduce the spaces

$$B^h = \{v \in M^3_{-1}(\tau_h); \quad v \textit{ vanishes on each } e \in \mathcal{E}_h, \}$$

and

$$N^h = V^h + B^h.$$

Let Π^0 denote the L^2-projection onto $M^0_{-1}(E^0_h)$, and P^0 the L^2-projection onto $M^0_{-1}(\tau_h)$. We have at our disposal two approximations of u, one, λ_h, is polynomial of degree k on each $e \in \mathcal{E}^0_h$, and the other, u_h, is polynomial of degree k in each element T of the triangulation τ_h. We use the two approximations to construct a unique function $\psi_h \in N^h$. More precisely, the following lemma is proved in [1].

Lemma 4.1 *Let k be an even integer, and $(\mathbf{p}_h, u_h, \lambda_h)$ the solution of (12). There exists a unique function $\psi_h \in N^h$ defined by*

$$P^0\psi_h = u_h,$$

and

$$\Pi^0\psi_h = \lambda_h.$$

Furthermore, the pair $(\mathbf{p}_h, \psi_h)$ solves the following problem:

$$\left\{\begin{array}{l} \text{find } (\mathbf{p}_h, \psi_h) \in R^h_{-1} \times N^h; \\ \displaystyle\int_\Omega \mathbf{C}\mathbf{p}_h \cdot \mathbf{q}\, dx - \sum_{T\in\tau_h} \int_T \mathbf{grad}\psi_h \cdot \mathbf{q}\, dx = 0 \quad \forall \mathbf{q} \in R^h_{-1}, \\ \displaystyle\sum_{T\in\tau_h} \int_T \mathbf{p}_h \cdot \mathbf{grad}\chi\, dx = -\int_\Omega (P^k f)\chi\, dx \qquad \forall \chi \in N^h. \end{array}\right. \tag{13}$$

Let now $P_{R,C}$ be the L^2-orthonormal projection onto R^h_{-1} with respect to the weighted L^2-inner product

$$(\mathbf{p}, \mathbf{q})_C = \int_\Omega \mathbf{C}\mathbf{p} \cdot \mathbf{q}\, dx.$$

From the first equation of problem (13), we have

$$\mathbf{p}_h = -P_{R,C}(\mathbf{A}\mathbf{grad}_h \psi_h), \tag{14}$$

where $(\mathbf{grad}_h\psi_h)|_T = \mathbf{grad}(\psi_h|_T) \quad \forall T \in \tau_h$. Therefore, it follows that problem (13) is equivalent to

$$\left\{\begin{array}{l} \text{find } \psi_h \in N^h; \\ \displaystyle c_h(\psi_h, \chi) = \int_\Omega (P^k f)\chi\, dx \quad \forall \chi \in N^h \end{array}\right. \tag{15}$$

where $c_h(\cdot,\cdot)$ is the positive semi-definite symmetric bilinear form on $H^1_0(\Omega) + N^h$ given by

$$c_h(\chi_1, \chi_2) = (P^k_{R,C}(\mathbf{A}\mathbf{grad}_h\chi_1), P^k_{R,C}(\mathbf{A}\mathbf{grad}_h\chi_2))_C. \tag{16}$$

We denote the corresponding nonconforming energy semi-norm on $H_0^1(\Omega) + N^h$ by

$$\| \psi \|_h = c_h(\psi, \psi)^{1/2}.$$

The steps from problem (11) to problem (15) can be viewed, at the matrix level, as a static condensation procedure to eliminate internal degrees of freedom. This strategy is general, and can be applied to linear systems of the form

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix},$$

whenever the matrix A corresponds to a bilinear continuous form on a space which does not have continuity requirement at the vertices, like the Brezzi-Douglas-Fortin-Marini (BDFM) spaces (cf. [5]), the Brezzi-Douglas-Marini (BDM) spaces (cf. [6]), and some mixed finite element spaces for the plate and elasticity equations.

Remark 4.1 *An important advantage of this nonconforming formulation is that it provides an approximate displacement field* $\overline{u}_h \in M_{-1}^1(\tau_h)$ *defined by*

$$\Pi^0(\psi_h - \overline{u}_h) = 0$$

which is more accurate than the approximation u_h *furnished by the original problem; see* [4].

In the following theorem we prove that the symmetric bilinear form $c_h(\cdot,\cdot)$ is uniformly positive definite on N^h.

Theorem 4.1 *There exists a positive constant* C *such that*

$$\|\psi\|_h \leq C \, \|\psi\|_{V^h} \quad \forall \psi \in H_0^1(\Omega) + N^h,$$

and

$$\|\psi\|_{V^h} \leq C \|\psi\|_h \quad \forall \psi \in N^h.$$

Proof. The hypotheses on $\mathbf{A}$ make the first inequality obvious. To prove the second inequality, observe that

$$\begin{aligned}
\| \psi \|_h &= \sup_{\mathbf{q} \in R_{-1}^h} \frac{|(P_{R,C}(\mathbf{A}\mathbf{grad}_h \psi), \mathbf{q})_C|}{(\mathbf{q}, \mathbf{q})_C^{1/2}} \\
&= \sup_{\mathbf{q} \in R_{-1}^h} \frac{|\int_\Omega \mathbf{grad}_h \psi \cdot \mathbf{q} \, dx|}{(\mathbf{q}, \mathbf{q})_C^{1/2}} \\
&\geq C \sup_{\mathbf{q} \in R_{-1}^h} \frac{|\int_\Omega \mathbf{grad}_h \psi \cdot \mathbf{q} \, dx|}{\| \mathbf{q} \|_{0,\Omega}} \\
&= C \, \| P_R(\mathbf{grad}_h \psi) \|_{0,\Omega}
\end{aligned}$$

where P_R is the orthogonal projection onto R^h_{-1} with respect to the usual inner product of $L^2(\Omega)$. It remains to show that there exists a positive constant C such that

$$\| P_R(\mathbf{grad}_h\psi) \|_{0,\Omega} \geq C \ \| \mathbf{grad}_h\psi \|_{0,\Omega} \quad \forall\psi \in N^h.$$

Let $\hat{T}$ be the unitary reference triangle $\{(x,y); x,y \geq 0, x+y \leq 1\}$ and e_i, $i=1,2,3$, its edges. Denote by $\boldsymbol{\nu}_i$ and $\mathbf{t}_i = (-\nu_i^2, \nu_i^1)$ the unit normal and tangent vector to e_i for $i=1,2,3$. Let $\psi = \varphi + b$ with $\varphi \in P^1(\hat{T})$ and $b \in xy(1-x-y)P^0(\hat{T})$. By definition the equation $P_R(\mathbf{grad}_h\psi) = 0$ on $\hat{T}$ means that

$$\int_{\hat{T}} \mathbf{grad}\psi \cdot \mathbf{q}\, dx = 0 \quad \forall \mathbf{q} \in R(\hat{T}). \tag{17}$$

Using Green's formula in (17), we obtain the relation

$$\int_{\hat{T}} \psi \operatorname{div} \mathbf{q}\, dx = \sum_{i=1}^{3} \int_{e_i} \varphi \mathbf{q} \cdot \boldsymbol{\nu}_i \, de_i \quad \forall \mathbf{q} \in R(\hat{T}). \tag{18}$$

Writing (18) for $\mathbf{q} \in (P^0(\hat{T}))^2 \subset R(\hat{T})$ we get

$$\sum_{i=1}^{3} \int_{e_i} \varphi \mathbf{q} \cdot \boldsymbol{\nu}_i \, de_i = 0 \quad \forall \mathbf{q} \in (P^0(\hat{T}))^2. \tag{19}$$

The restriction of φ to each e_i is a polynomial of degree ≤ 1 which can be written as $\varphi|_{e_i} = \alpha_i L_i + \beta_i$ for $i = 1,2,3$, where α_i,β_i are two constants and L_i is the Legendre polynomial of degree 1 on e_i. Then equation (19) gives

$$\sum_{i=1}^{3} \beta_i \int_{e_i} \mathbf{q} \cdot \boldsymbol{\nu}_i \, de_i = 0 \quad \forall \mathbf{q} \in (P^0(\hat{T}))^2.$$

In this equation, we put $\mathbf{q} = \mathbf{t}_1$ and $\mathbf{q} = \mathbf{t}_2$ successively to deduce that $\beta_1 = \beta_2 = \beta_3$. Denote this common constant β. The above shows that the function $\psi - \beta$ is a multiple of the Legendre polynomial of degree 1 on each e_i. Thus $\psi - \beta$ takes opposite values at the end points of each e_i. The Legendre polynomials do not vanish at the endpoints of the interval. Since $\psi - \beta$ is continuous on $\hat{T}$, this implies the equality $\psi - \beta = 0$ on $\partial\hat{T}$. Therefore, $\psi - \beta = xy(1-x-y)z$ where $z \in P^0(\hat{T})$. Now, using once again (17) we have

$$\int_{\hat{T}} (\psi - \beta) \operatorname{div} \mathbf{q}\, dx = 0 \quad \forall \mathbf{q} \in R(\hat{T}), \tag{20}$$

and choosing in (20) $\mathbf{q} \in R(\hat{T})$ such that $\operatorname{div} \mathbf{q} = z$ we conclude that $\psi = \beta$ on $\hat{T}$.

The two norms

$$\psi \longrightarrow \| P_R(\mathbf{grad}\psi) \|_{0,\hat{T}}$$

and

$$\psi \longrightarrow \| \mathbf{grad}\psi \|_{0,\hat{T}}$$

are equivalent on the quotient space $(P^1(\hat{T}) + xy(1-x-y)P^0(\hat{T}))/P^0(\hat{T})$, so there exists a constant C such that

$$\| P_R(\mathbf{grad}\psi) \|_{0,\hat{T}} \geq C \ \| \mathbf{grad}\psi \|_{0,\hat{T}} .$$

By definition

$$\| P_R(\mathbf{grad}\psi) \|_{0,T} = \sup_{\mathbf{q}\in R(\hat{T})} \frac{|\int_T \mathbf{grad}\psi \cdot \mathbf{q}\, dx|}{\| \mathbf{q} \|_{0,T}}.$$

We deduce by standard scaling arguments that there exists a constant C independent of h such that

$$\| P_R(\mathbf{grad}\psi) \|_{0,T} \geq C \ \| \mathbf{grad}\psi \|_{0,T} \quad \forall \psi \in P^1(T) + xy(1-x-y)P^0(T),$$

for all $T \in \tau_h$. Summing over all the triangles we obtain the result. □

Remark 4.2 *A generalization of the last theorem to the case of greater order finite elements is given in [18].*

The following lemma indicates that estimates on N^h can be done on the components V^h and B^h separately.

Lemma 4.2 *Let $\psi = \varphi + b$ be an arbitrary element of N^h where $\varphi \in V^h$ and $b \in B^h$. Then, there exists a positive constant C such that*

$$C(\|\varphi\|_h + \|b\|_h) \quad \leq \|\psi\|_h \leq \|\varphi\|_h + \|b\|_h.$$

Proof. The upper bound of the lemma follows from the triangular inequality. To get the lower bound one can use Theorem 4.1 to see that it is equivalent to state the inequality

$$C(\|\varphi\|_{V^h} + \|b\|_{V^h}) \leq \|\psi\|_{V^h}.$$

Let $\hat{T}$ be the reference triangle. The norms

$$\psi \longrightarrow \|\mathbf{grad}\psi\|_{0,\hat{T}},$$

and

$$\psi \longrightarrow \|\mathbf{grad}\varphi\|_{0,\hat{T}} + \|\mathbf{grad}b\|_{0,\hat{T}},$$

are equivalent on $(P^1(\hat{T}) + xy(1-x-y)P^0(\hat{T}))/P^0(\hat{T})$. By a change of variable and summing over all the triangles we obtain the result. □

We associate to each Ω_{ij} the subspace

$$N^h_{ij} = \{\psi \in N^h; \quad \text{supp}(\psi) \subset \overline{\Omega}_{ij}\}.$$

Note that

$$N^h_{ij} = V^h_{ij} + B^h_{ij}$$

where

$$b^h_{ij} = B^h \cap H^1_0(\Omega_{ij}).$$

Theorem 4.2 *There exists a constant C such that for each $\psi \in N^h$ there exists a decomposition*

$$\psi = \psi_0 + \sum_{ij} \psi_{ij}$$

with $\psi_0 \in M_0^1(\tau_H)$ and $\psi_{ij} \in N_{ij}^h$ satisfying

$$\|\psi_0\|_h^2 + \sum_{ij} \|\psi_{ij}\|_h^2 \leq C(1 + \log(\frac{H}{h}))^2 \|\psi\|_h^2.$$

Proof. Let $\psi \in N^h$, $\psi = \varphi + b$ with $\varphi \in V^h$ and $b \in B^h$. By Theorem 3.1 we know that there exists a decomposition

$$\varphi = \varphi_0 + \sum_{ij} \varphi_{ij}$$

with $\varphi_0 \in M_0^1(\tau_H)$ and $\varphi_{ij} \in V_{ij}^h$ satisfying

$$\|\psi_0\|_{V^h}^2 + \sum_{ij} \|\psi_{ij}\|_{V^h}^2 \leq C(1 + \log \frac{H}{h})^2 \|\psi\|_{V^h}^2.$$

Now , we decompose b as follows

$$b = \sum_{ij} b_{ij}$$

with $b_{ij} \in B_{ij}^h$ defined by $b_{ij} = \alpha_i b|_{\Omega_i}$ in Ω_i and $\alpha_j b|_{\Omega_j}$ in Ω_j. It is Straightforward that $b = \sum_{ij} b_{ij}$ and

$$\sum_{ij} \|b_{ij}\|_{V^h}^2 = \|b\|_{V^h}^2.$$

We take $\psi_0 = \varphi_0$, $\psi_{ij} = \varphi_{ij} + b_{ij}$ and conclude by Lemma 4.2.

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