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# *Avoiding maximal parabolic subgroups of $S_k$*

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We find an explicit expression for the generating function of the number of permutations in $S_n$ avoiding a subgroup of $S_k$ generated by all but one simple transpositions. The generating function turns out to be rational, and its denominator is a rook polynomial for a rectangular board.

**Keywords:** permutations, forbidden patterns, parabolic subgroups, Laguerre polynomials, rook polynomials

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## 1 Introduction and Main Result

Let $[p] = \{1,\dots,p\}$ denote a totally ordered alphabet on $p$ letters, and let $\alpha = (\alpha_1,\dots,\alpha_m) \in [p_1]^m$, $\beta = (\beta_1,\dots,\beta_m) \in [p_2]^m$. We say that $\alpha$ is *order-isomorphic* to $\beta$ if for all $1 \le i < j \le m$ one has $\alpha_i < \alpha_j$ if and only if $\beta_i < \beta_j$. For two permutations $\pi \in S_n$ and $\tau \in S_k$, an *occurrence* of $\tau$ in $\pi$ is a subsequence $1 \le i_1 < i_2 < \dots < i_k \le n$ such that $(\pi_{i_1},\dots,\pi_{i_k})$ is order-isomorphic to $\tau$; in such a context $\tau$ is usually called the *pattern*. We say that $\pi$ *avoids* $\tau$, or is $\tau$-*avoiding*, if there is no occurrence of $\tau$ in $\pi$. Pattern avoidance proved to be a useful language in a variety of seemingly unrelated problems, from stack sorting [Kn, Ch. 2.2.1] to singularities of Schubert varieties [LS]. A natural generalization of single pattern avoidance is *subset avoidance*; that is, we say that $\pi \in S_n$ avoids a subset $T \subset S_k$ if $\pi$ avoids any $\tau \in T$. The set of all permutations in $S_n$ avoiding $T \subset S_k$ is denoted $S_n(T)$. A complete study of subset avoidance for the case $k = 3$ is carried out in [SS]. For $k > 3$ the situation becomes more complicated, as the number of possible cases grows rapidly. Recently, several authors have considered the case of general $k$ when $T$ has some nice algebraic properties. Paper [BDPP] treats the case when $T$ is the centralizer of $k-1$ and $k$ under the natural action of $S_k$ on $[k]$ (see also Sec. 3 for more detail). In [AR], $T$ is a Kazhdan–Lusztig cell of $S_k$, or, equivalently, the Knuth equivalence class (see [St, vol. 2, Ch. A1]). In this paper we consider the case when $T$ is a maximal parabolic subgroup of $S_k$.

Let $s_i$ denote the simple transposition interchanging $i$ and $i+1$. Recall that a subgroup of $S_k$ is called *parabolic* if it is generated by $s_{i_1},\dots,s_{i_r}$. A parabolic subgroup of $S_k$ is called *maximal* if the number of its generators equals $k-2$. We denote by $P_{l,m}$ the (maximal) parabolic subgroup of $S_{l+m}$ generated by $s_1,\dots,s_{l-1},s_{l+1},\dots,s_{l+m-1}$, and by $f_{l,m}(n)$ the number of permutations in $S_n$ avoiding all the patterns in $P_{l,m}$. In this note we find an explicit expression for the generating function of the sequence $\{f_{l,m}(n)\}$.

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To be more precise, we prove the following more general result. Let us denote $\sigma = s_1 s_2 \dots s_{k-1}$, that is, $\sigma = (2,3,\dots,k,1)$ (written in one-line notation), and let $a$ be an integer, $0 \le a \le k-1$ (here and in what follows $k = l+m$). We denote by $f^a_{l,m}(n)$ the number of permutations in $S_n$ avoiding the left coset $\sigma^a P_{l,m}$; in particular, $f^0_{l,m}(n)$ coincides with $f_{l,m}(n)$. Let $F^a_{l,m}(x)$ denote the generating function of $\{f^a_{l,m}(n)\}$,

$$F^a_{l,m}(x) = \sum_{n\ge 0} f^a_{l,m}(n) x^n.$$

Recall that the *Laguerre polynomial* $L^\alpha_n(x)$ is given by

$$L^\alpha_n(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n}\left(e^{-x} x^{n+\alpha}\right),$$

and the *rook polynomial* of the rectangular $s \times t$ board is given by

$$R_{s,t}(x) = s! x^s L^{t-s}_s(-x^{-1})$$

for $s \le t$ and by $R_{s,t}(x) = R_{t,s}(x)$ otherwise (see [Ri, Ch. 7.4]).

**Main Theorem.** *Let $\lambda = \min\{l,m\}$, $\mu = \max\{l,m\}$, then*

$$F^a_{l,m}(x) R_{l,m}(-x) = \sum_{r=0}^{\lambda-1} x^r r! \sum_{j=0}^{r} (-1)^j \frac{\binom{l}{j}\binom{m}{j}}{\binom{r}{j}} + (-1)^\lambda x^\lambda \lambda! \sum_{r=0}^{\mu-\lambda-1} x^r r! \binom{\mu-r-1}{\lambda},$$

*or, equivalently,*

$$F^a_{l,m}(x) = \sum_{r=0}^{k-1} x^r r! - \frac{(-1)^\lambda x^\mu}{\lambda! L^{\mu-\lambda}_\lambda(x^{-1})} \sum_{r=0}^{\lambda-1} (k+r)! x^r \sum_{j=r+1}^{\lambda} (-1)^j \frac{\binom{l}{j}\binom{m}{j}}{\binom{k+r}{j}},$$

*where $k = l+m = \lambda+\mu$.*

The proof of the Main Theorem is presented in the next section.

As a corollary we immediately get the following result (see [Ma, Theorem 1]).

**Corollary 1** *Let $0 \le a \le k-1$, then*

$$f^a_{1,k-1}(n) = \begin{cases} (k-2)!(k-1)^{n+2-k} & \text{for } n \ge k \\ n! & \text{for } n < k. \end{cases}$$

*Proof.* Since $R_{1,k-1} = 1 + (k-1)x$, the Main Theorem implies

$$F^a_{1,k-1}(x) = \frac{1 - \sum_{r=0}^{k-3} x^{r+1}(k-r-2)r!}{1-(k-1)x} = \frac{x^{k-2}(k-2)!}{1-(k-1)x} + \sum_{r=0}^{k-3} x^r r!,$$

and the result follows. □

Another immediate corollary of the Main Theorem gives the asymptotics for $f^a_{l,m}(n)$ as $n \to \infty$.

**Corollary 1.2.** $f^a_{l,m}(n) \sim c\gamma^n$, *where $c$ is a constant depending on $l$ and $m$, and $\gamma$ is the maximal root of $L^{\mu-\lambda}_\lambda$; in particular, $\gamma \le k-2+\sqrt{1+4(l-1)(m-1)}$.*

*Proof.* Follows from standard results in the theory of rational generating functions (see e.g. [St, vol. 1, Ch. 4]) and the fact that all the roots of Laguerre polynomials are simple (see [Sz, Ch. 3.3]). The upper bound on $\gamma$ is obtained in [IL]. □

## 2 Proofs

First of all, we make the following simple, though useful observation.

**Lemma 2** *For any natural a, l, m, n such that* $1 \le a \le l+m-1$ *one has* $f^a_{l,m}(n) = f^{m+l-a}_{m,l}(n)$.

*Proof*. Denote by $\rho_n$ and $\kappa_n$ the involutions $S_n \to S_n$ that take $(i_1,i_2,\dots,i_k)$ to $(i_k,\dots,i_2,i_1)$ (*reversal*) to $(n+1-i_1, n+1-i_2,\dots,n+1-i_n)$ (*complement*), respectively. It is easy to see that for any $T \subset S_k$, the involutions $\rho_n$ and $\kappa_n$ provide natural bijections between the sets $S_n(T)$ and $S_n(\rho_k T)$, and between $S_n(T)$ and $S_n(\kappa_k T)$, respectively. It remains to note that $\rho_k\kappa_k\sigma^a P_{l,m} = \sigma^{l+m-a}P_{m,l}$. □

From now on we assume that $a \ge 0$, $l \ge 1$, $m \ge 1$ are fixed, and denote $b = n-m+a$. It follows from Lemma 2 that we may assume that $a \le m$, and hence $b \le n$. This means, in other words, that $\tau \in S_k$ belongs to $\sigma^a P_{l,m}$ if and only if $(\tau_1,\dots,\tau_l)$ is a permutation of the numbers $a+1,\dots,a+l$. In what follows we usually omit the indices $a$, $l$, $m$ whenever appropriate; for example, instead of $f^a_{l,m}(n)$ we write just $f(n)$.

For any $n \ge k$ and any $d$ such that $1 \le d \le n$, we denote by $g_n(i_1,\dots,i_d) = g^a_{n;l,m}(i_1,\dots,i_d)$ the number of permutations $\pi \in S_n(\sigma^a P_{l,m})$ such that $\pi_j = i_j$ for $j = 1,\dots,d$. It is natural to extend $g_n$ to the case $d = 0$ by setting $g_n(\emptyset) = f(n)$.

The following properties of the numbers $g_n(i_1,\dots,i_d)$ can be deduced easily from the definitions.

**Lemma 3**

*(i) Let* $n \ge k$ *and* $1 \le i \le n$, *then*

$$g_n(\dots,i,\dots,i,\dots) = 0.$$

*(ii) Let* $n \ge k$ *and* $a+1 \le i_j \le b$ *for* $j = 1,\dots,l$, *then*

$$g_n(i_1,\dots,i_l) = 0.$$

*(iii) Let* $n \ge k$, $1 \le r \le d \le l$, *and* $a+1 \le i_j \le b$ *for* $j = 1,\dots,d$, $j \ne r$, *then*

$$g_n(i_1,\dots,i_d) = \begin{cases} g_{n-1}(i_1-1,\dots,i_{r-1}-1,i_{r+1}-1,\dots,i_d-1) & \text{if } 1 \le i_r \le a \\ g_{n-1}(i_1,\dots,i_{r-1},i_{r+1},\dots,i_d) & \text{if } b+1 \le i_r \le n. \end{cases}$$

*Proof*. Property (i) is evident. Let us prove (ii). By (i), we may assume that the numbers $i_1,\dots,i_l$ are distinct. Take an arbitrary $\pi \in S_n$ such that $\pi_j = i_j$ for $j = 1,\dots,l$. Evidently, for any $r \le a$ there exists a position $j_r > l$ such that $\pi_{j_r} = r$; the same is true for any $r \ge b+1$. Therefore, the restriction of $\pi$ to the positions $1,2,\dots,l,j_1,j_2,\dots,j_a,j_{b+1},j_{b+2},\dots,j_n$ (in the proper order) gives an occurrence of $\tau \in \sigma^a P_{l,m}$ in $\pi$. Hence, $\pi \notin S_n(\sigma^a P_{l,m})$, which means that $g_n(i_1,\dots,i_l) = 0$.

To prove (iii), assume first that $1 \le i_r \le a$. Let $\pi \in S_n$ and $\pi_j = i_j$ for $j = 1,\dots,d$. We define $\pi^* \in S_{n-1}$ by

$$\pi^*_j = \begin{cases} \pi_j - 1 & \text{for } 1 \le j \le r-1, \\ \pi_{j+1} - 1 & \text{for } j \ge r \text{ and } \pi_{j+1} > i_r, \\ \pi_{j+1} & \text{for } j \ge r \text{ and } \pi_{j+1} < i_r. \end{cases} \tag{1}$$

We claim that $\pi \in S_n(\sigma^a P_{l,m})$ if and only if $\pi^* \in S_{n-1}(\sigma^a P_{l,m})$. Indeed, the only if part is trivial, since any occurrence of $\tau \in \sigma^a P_{l,m}$ in $\pi^*$ immediately gives rise to an occurrence of $\tau$ in $\pi$. Conversely, any occurrence of $\tau$ in $\pi$ that does not include $i_r$ gives rise to an occurrence of $\tau$ in $\pi^*$. Assume that there exists an occurrence of $\tau$ in $\pi$ that includes $i_r$. Since $r \le d \le l$, this occurrence of $\tau$ contains $a$ entries that are situated to the right of $i_r$ and are strictly less than $i_r$. However, the whole $\pi$ contains only $a-1$ such entries, a contradiction. It now follows from (1) that property (iii) holds for $1 \le i_r \le a$. The case $b+1 \le i_r \le n$ is treated similarly with the help of the transformation $(\pi \in S_n) \mapsto (\pi^\circ \in S_{n-1})$ given by

$$\pi_j^\circ = \begin{cases} \pi_j & \text{for } 1 \le j \le r-1, \\ \pi_{j+1} - 1 & \text{for } j \ge r \text{ and } \pi_{j+1} > i_r, \\ \pi_{j+1} & \text{for } j \ge r \text{ and } \pi_{j+1} < i_r. \end{cases}$$

□

Now we introduce the quantity that plays the crucial role in the proof of the Main Theorem. For $n \ge k$ and $1 \le d \le l$ we put

$$A(n,d) = A_{l,m}^a(n,d) = \sum_{i_1,\ldots,i_d=a+1}^{b} g_n(i_1,\ldots,i_d).$$

As before, this definition is extended to the case $d=0$ by setting

$$A(n,0) = g_n(\emptyset) = f(n).$$

**Theorem 4** *Let $n \ge k+1$ and $1 \le d \le l-1$, then*

$$A(n,d+1) = A(n,d) - (m-d)A(n-1,d) - dA(n-1,d-1). \tag{2}$$

*Proof*. First of all, we introduce two auxiliary sums:

$$\begin{aligned} B(n,d) &= B_{l,m}^a(n,d) = \sum_{i_1,\ldots,i_d=a+1}^{b+1} g_n(i_1,\ldots,i_d), \\ C(n,d) &= C_{l,m}^a(n,d) = \sum_{i_1,\ldots,i_d=a}^{b} g_n(i_1,\ldots,i_d), \end{aligned}$$

where $b = n-m+a$; once again, $B(n,0) = C(n,0) = f(n)$.

Let us prove three simple identities relating together the sequences $\{A(n,d)\}$, $\{B(n,d)\}$, $\{C(n,d)\}$.

**Lemma 5** *Let $n \ge k$ and $1 \le d \le l$, then:*

$$\begin{gathered} A(n,d) = A(n,d-1) - (m-a)B(n-1,d-1) - aC(n-1,d-1), \\ (m-a)A(n,d) = (m-a)B(n,d) - (m-a)dB(n-1,d-1), \\ aA(n,d) = aC(n,d) - adC(n-1,d-1). \end{gathered}$$

*Proof*. To prove the first identity, observe that by definitions and Lemma 3(iii) for the case $r=d$, one has

$$A(n,d-1) - A(n,d) = \sum_{i_1,\ldots,i_{d-1}=a+1}^{b} \sum_{i_d=1}^{n} g_n(i_1,\ldots,i_d) - A(n,d)$$

$$
\begin{aligned}
&= \sum_{i_1,\ldots,i_{d-1}=a+1}^{b} \left( \sum_{i_d=1}^{a} g_n(i_1,\ldots,i_d) + \sum_{i_d=b+1}^{n} g_n(i_1,\ldots,i_d) \right) \\
&= \sum_{i_1,\ldots,i_{d-1}=a+1}^{b} \big( a g_{n-1}(i_1-1,\ldots,i_{d-1}-1) + (m-a) g_{n-1}(i_1,\ldots,i_{d-1}) \big) \\
&= aC(n-1,d-1) + (m-a)B(n-1,d-1),
\end{aligned}
$$

and the result follows.

The second identity is trivial for $a = m$, so assume that $0 \le a \le m-1$ and observe that by definitions and Lemma 3(ii) and (iii), one has

$$
\begin{aligned}
B(n,d) &= \sum_{i_1,\ldots,i_d=a+1}^{b} g_n(i_1,\ldots,i_d) + \sum_{j=1}^{d} \left( \sum_{i_1,\ldots,\hat{\imath}_j,\ldots,i_d=a+1}^{b} g_n(i_1,\ldots,i_{j-1},b+1,i_{j+1}\ldots,i_d) \right) \\
&= A(n,d) + \sum_{j=1}^{d} \left( \sum_{i_1,\ldots,\hat{\imath}_j,\ldots,i_d=a+1}^{b} g_{n-1}(i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_d) \right) \\
&= A(n,d) + dB(n-1,d-1),
\end{aligned}
$$

and the result follows.

Finally, the third identity is trivial for $a = 0$, so assume that $1 \le a \le m$ and observe that by definitions and Lemma 3(ii) and (iii), one has

$$
\begin{aligned}
C(n,d) &= \sum_{i_1,\ldots,i_d=a+1}^{b} g_n(i_1,\ldots,i_d) + \sum_{j=1}^{d} \left( \sum_{i_1,\ldots,\hat{\imath}_j,\ldots,i_d=a+1}^{b} g_n(i_1,\ldots,i_{j-1},a,i_{j+1}\ldots,i_d) \right) \\
&= A(n,d) + \sum_{j=1}^{d} \left( \sum_{i_1,\ldots,\hat{\imath}_j,\ldots,i_d=a+1}^{b} g_{n-1}(i_1-1,\ldots,i_{j-1}-1,i_{j+1}-1,\ldots,i_d-1) \right) \\
&= A(n,d) + dC(n-1,d-1),
\end{aligned}
$$

and the result follows. □

Now we can complete the proof of Theorem 4. Indeed, using twice the first identity of Lemma 5, one gets

$$
\begin{aligned}
A(n,d+1) &= A(n,d) - (m-a)B(n-1,d) - aC(n-1,d-1), \\
dA(n-1,d) &= dA(n-1,d-1) - d(m-a)B(n-2,d-1) - daC(n-2,d-1).
\end{aligned}
$$

Next, the other two identities of Lemma 5 imply

$$
\begin{aligned}
&A(n,d+1) - dA(n-1,d) = A(n,d) - dA(n-1,d-1) \\
&\quad - \big((m-a)B(n-1,d) - (m-a)dB(n-2,d-1)\big) - \big(aC(n-1,d) - adC(n-2,d-1)\big) \\
&\quad = A(n,d) - dA(n-1,d-1) - (m-a)A(n-1,d) - aA(n-1,d),
\end{aligned}
$$

and the result follows. □

The next result relates the sequence $\{A(n,d)\}$ to the sequence $\{f(n)\}$.

**Theorem 6** *Let $n \geq k$ and $1 \leq d \leq l$, then*

$$A(n,d) = \sum_{j=0}^{d} (-1)^j j! \binom{m}{j} \binom{d}{j} f(n-j).$$

*Proof*. Let $D(n,d) = D^a_{l,m}(n,d)$ denote the right hand side of the above identity. We claim that for $n \geq k+1$ and $1 \leq d \leq l-1$, $D(n,d)$ satisfies the same relation (2) as $A(n,d)$ does. Indeed,

$$\begin{aligned} D(n-1,d) &= \sum_{j=0}^{d} (-1)^j j! \binom{m}{j} \binom{d}{j} f(n-1-j) \\ &= -\sum_{j=1}^{d} (-1)^j (j-1)! \binom{m}{j-1} \binom{d}{j-1} f(n-j) + (-1)^d d! \binom{m}{d} f(n-d-1), \end{aligned}$$

and

$$\begin{aligned} D(n-1,d-1) &= \sum_{j=0}^{d-1} (-1)^j j! \binom{m}{j} \binom{d-1}{j} f(n-1-j) \\ &= -\sum_{j=1}^{d} (-1)^j (j-1)! \binom{m}{j-1} \binom{d-1}{j-1} f(n-j), \end{aligned}$$

and hence

$$\begin{aligned} &D(n,d) - (m-d)D(n-1,d) - dD(n-1,d-1) = f(n) + (m-d)(-1)^{d+1} d! \binom{m}{d} f(n-d-1) \\ &\quad + \sum_{j=1}^{d} (-1)^j j! \left( \binom{m}{j} \binom{d}{j} + \frac{m-d}{j} \binom{m}{j-1} \binom{d}{j-1} + \frac{d}{j} \binom{m}{j-1} \binom{d-1}{j-1} \right) f(n-j) \\ &\quad = f(n) + \sum_{j=1}^{d} (-1)^j j! \binom{m}{j} \binom{d+1}{j} f(n-j) + (-1)^{d+1} (d+1)! \binom{m}{d+1} f(n-d-1) = D(n,d+1). \end{aligned}$$

It follows that $D(n,d)$ (as well as $A(n,d)$) are defined uniquely for $n \geq k$ and $1 \leq l \leq d$ by initial values $D(k,d)$, $D(n,0)$, and $D(n,1)$ ($A(k,d)$, $A(n,0)$, and $A(n,1)$, respectively). It is easy to see that for $n \geq k$ one has $A(n,0) = D(n,0) = f(n)$. Next, the first identity of Lemma 5 for $d = 1$ gives

$$A(n,1) = A(n,0) - (m-a)B(n-1,0) - aC(n-1,0) = f(n) - mf(n-1) \quad \text{for } n \geq k.$$

On the other hand, by definition,

$$D(n,1) = f(n) - mf(n-1) \quad \text{for } n \geq k,$$

and hence $A(n,1) = D(n,1)$. Finally, a simple combinatorial argument shows that

$$A(k,d) = d! \binom{l}{d} (k-d)! - l!m! \quad \text{for } 1 \leq d \leq l.$$

On the other hand,

$$D(k,d) = \sum_{j=0}^{d} (-1)^j j! \binom{m}{j} \binom{d}{j} (k-j)! - l!m!,$$

since $f(r) = r!$ for $1 \le r \le k-1$ and $f(k) = k! - l!m!$. To prove $A(k,d) = D(k,d)$ it remains to check that

$$\sum_{j=0}^{d} (-1)^j j! \binom{m}{j} \binom{d}{j} (k-j)! = d! \binom{l}{d} (k-d)!,$$

which follows from Lemma 7 below. □

Finally, we are ready to prove the Main Theorem stated in Sec. 1. First of all, by Lemma 3(ii), $A(n,l) = 0$ for $n \ge k$. Hence, by Theorem 6,

$$\sum_{j=0}^{l} (-1)^j j! \binom{m}{j} \binom{l}{j} f(n-j) = 0 \quad \text{for } n \ge k,$$

or, equivalently,

$$\sum_{j=0}^{l} (-1)^j j! \binom{m}{j} \binom{l}{j} x^j f(n-j) x^{n-j} = 0 \quad \text{for } n \ge k.$$

As was already mentioned, $f(r) = r!$ for $1 \le r \le k-1$, therefore, summing over $n \ge k$ yields

$$\sum_{j=0}^{l} (-1)^j j! \binom{m}{j} \binom{l}{j} x^j \left( F^a_{l,m}(x) - \sum_{i=0}^{k-j-1} x^i i! \right) = 0. \tag{3}$$

Recall that the rook polynomial of the rectangular $s \times t$ board, $s \le t$, satisfies relation

$$R_{s,t}(x) = \sum_{j=0}^{s} j! \binom{t}{j} \binom{s}{j} x^j$$

(see [Ri, Ch. 7.4]). Hence, (3) is equivalent to

$$F^a_{l,m}(x) R_{\lambda,\mu}(-x) = \sum_{j=0}^{l} (-1)^j j! \binom{m}{j} \binom{l}{j} x^j \sum_{i=0}^{k-j-1} x^i i! = \sum_{r=0}^{k-1} x^r r! \sum_{j=0}^{r} (-1)^j \frac{\binom{l}{j}\binom{m}{j}}{\binom{r}{j}}.$$

Let us divide the external sum in the above expression into three parts: the sum from $r = 0$ to $\lambda - 1$, the sum from $r = \lambda$ to $\mu - 1$, and the sum from $r = \mu$ to $k-1$. By Lemma 7 below, the third sum vanishes, while the second sum is equal to

$$\sum_{r=\lambda}^{\mu-1} x^r r! (-1)^{\lambda} \frac{(r-\lambda)!(k-r-1)!}{(\mu-r-1)!r!},$$

and the first expression of the Main Theorem follows. The second expression is obtained easily from (3) and relation between rook polynomials and Laguerre polynomials given in Sec. 1. □

It remains to prove the following technical result, which is apparently known; however, we failed to find a reference to its proof, and decided to present a short proof inspired by the brilliant book [PWZ].

**Lemma 7** *Let* $1 \le s \le t$ *and let*

$$M(s,t)=\sum_{i=0}^{s}(-1)^i\frac{\binom{s}{i}\binom{t}{i}}{\binom{n}{i}}.$$

*Then:*

$$M(s,t)=\begin{cases}\frac{\binom{n-t}{s}}{\binom{n}{s}} & \text{if } n\ge s+t,\\ 0 & \text{if } t\le n\le s+t-1,\\ (-1)^s\frac{\binom{s+t-n-1}{s}}{\binom{n}{s}} & \text{if } s\le n\le t-1.\end{cases}$$

*Proof*. Direct check reveals that $M(s,t)$ is a hypergeometric series; to be more precise,

$$M(s,t)={}_2F_1\left[{-t,-s \atop -n};1\right].$$

Since $-s$ is a nonpositive integer, the Gauss formula applies (see [PWZ, Ch. 3.5]), and we get

$$M(s,t)=\lim_{z\to n}\frac{\Gamma(-z+t+s)\Gamma(-z)}{\Gamma(t-z)\Gamma(s-z)}.$$

Recall that

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}. \tag{4}$$

If $n\ge s+t$, we apply (4) for $x=-z+t+s$, $x=t-z$, $x=s-z$, $x=-z$, and get

$$M(s,t)=-\frac{\Gamma(n-t+1)\Gamma(n-s+1)}{\Gamma(n-t-s+1)\Gamma(n+1)}\lim_{z\to n}\frac{\sin\pi(t-z)\sin\pi(s-z)}{\sin\pi z\sin\pi(t+s-z)}=\frac{\binom{n-t}{s}}{\binom{n}{s}}.$$

If $t\le n\le s+t-1$, we apply (4) for $x=t-z$, $x=s-z$, $x=-z$, and get

$$M(s,t)=-\frac{\Gamma(n-t+1)\Gamma(n-s+1)\Gamma(s+t-n)}{\Gamma(n+1)}\lim_{z\to n}\frac{\sin\pi(t-z)\sin\pi(s-z)}{\sin\pi z}=0.$$

Finally, if $s\le n\le t-1$, we apply (4) for $x=s-z$, $x=-z$, and get

$$M(s,t)=-\frac{\Gamma(s+t-n)\Gamma(n-s+1)}{\Gamma(t-n)\Gamma(n+1)}\lim_{z\to n}\frac{\sin\pi(s-z)}{\sin\pi z}=(-1)^s\frac{\binom{s+t-n-1}{s}}{\binom{n}{s}}.$$

□

# 3 Concluding remarks

Observe first, that according to the Main Theorem, $F^a_{l,m}(x)$ does not depend on $a$; in other words, $|S_n(P_{l,m})|=|S_n(\sigma^a P_{l,m})|$ for any $a$. We obtained this fact as a consequence of lengthy computations. A natural question would be to find a bijection between $S_n(P_{l,m})$ and $S_n(\sigma^a P_{l,m})$ that explains this phenomenon.

Second, it is well known that rook polynomials (or the corresponding Laguerre polynomials) are related to permutations with restricted positions, see [Ri, Ch.7,8]. Laguerre polynomials also arise in a natural way in the study of generalized derangements (see [FZ] and references therein). It is tempting to find a combinatorial relation between permutations with restricted positions and permutations avoiding maximal parabolic subgroups, which could explain the occurrence of Laguerre polynomials in the latter context.

Finally, one can consider permutations avoiding nonmaximal parabolic subgroups of $S_k$. The first natural step would be to treat the case of subgroups generated by $k-3$ simple transpositions. It is convenient to denote by $P_{l_1,l_2,l_3}$ (with $l_1+l_2+l_3=k$) the subgroup of $S_k$ generated by all the simple transpositions except for $s_{l_1}$ and $s_{l_1+l_2}$; further on, we set $f_{l_1,l_2,l_3}(n)=|S_n(P_{l_1,l_2,l_3})|$, and $F_{l_1,l_2,l_3}(x)=\sum_{n\geq 0} f_{l_1,l_2,l_3}(n)x^n$. It is easy to see that $F_{l_1,l_2,l_3}(x)=F_{l_3,l_2,l_1}(x)$, so one can assume that $l_1\leq l_3$. This said, the main result of [BDPP] can be formulated as follows: let $k\geq 3$, then

$$F_{1,1,k-2}(x)=\sum_{r=1}^{k-3} x^r r! + \frac{(k-3)!x^{k-4}}{2}\left(1-(k-1)x-\sqrt{1-2(k-1)x+(k-3)^2x^2}\right).$$

To the best of our knowledge, this is the only known instance of $F_{l_1,l_2,l_3}(x)$. It is worth noting that even in this, simplest case of nonmaximal parabolic subgroup, the generating function is no longer rational.

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