A Boundary Rigidity Problem for Holomorphic Mappings

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Abstract

Several families of complex geodesics of $D^2 = \{z = (z_1, z_2) \in \mathcal{C}^2 : |z_1| + |z_2| < 1\}$ touching given boundary points are explicitly studied and the corresponding holomorphic retracts in the sense of Lempert [12] are founded. A careful investigation of the holomorphic extendibility of the Lempert–type retracts to the boundary of D^2 and a few results on their "uniqueness" allows the proof of a Cartan–type rigidity theorem at the boundary for holomorphic mappings of the domain D^2. Analogous rigidity results are obtained for special points of the boundary of D^n, $n > 2$.

1 Introduction

One of the main features of the theory of holomorphic maps is the close connection between analytic properties of maps and geometric properties of the domains on which they are defined. The classical Schwarz and Schwarz-Pick Lemmas are among the most significant examples in this setting: they establish that any holomorphic map f of the open unit disc Δ of $\mathcal{C}$ into itself is a contraction for the Poincaré metric, that is, for any $z_0 \in \Delta$, f maps any Poincaré disc centered at z_0 into a Poincaré disc centered at $f(z_0)$ with the same radius.

In particular the Schwarz and Schwarz-Pick Lemmas establish the following "rigidity result": if $f : \Delta \to \Delta$ is holomorphic and if $z_0 \in \Delta$ is such that $f(z_0) = z_0$, then $|f'(z_0)| \leq 1$; moreover $f'(z_0) = 1$ if, and only if, f is the identity on Δ.

In the case of several complex variables, a generalization of the Schwarz Lemma appears in a classical theorem proved by H. Cartan, according to which, if $\Omega \subset \mathcal{C}^n$ is a bounded domain, if $f : \Omega \to \Omega$ is a holomorphic map and if $z_0 \in \Omega$ is such that $f(z_0) = z_0$, then $|det(df_{z_0})| \leq 1$; moreover df_{z_0} is the identity if, and only if, f is the identity on Ω.

The investigation on the behaviour of the Poincaré metric at the boundary of the unit disc Δ of $\mathcal{C}$, carried out by using geometric tools such as horospheres and Stolz regions, leads to the Julia Lemma and the Wolff Lemma which establish Schwarz-type results at the boundary of Δ. In fact it turns out that if $f : \Delta \to \Delta$ is holomorphic and without fixed points, then there exists a unique $\tau \in \partial\Delta$ such that $f(\tau) = \tau$ and that $0 < f'(\tau) \leq 1$ (here $f(\tau)$ and $f'(\tau)$ are defined as non-tangential limits). For such a τ - the Wolff point of f - and for any $R > 0$, the horosphere of radius R centered

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at τ is mapped into itself. However the rigidity result stated by the classical Schwarz Lemma does not extend in the same form to the case of a holomorphic map having its Wolff point $\tau \in \partial\Delta$; take for example the map $g(z) = \dfrac{1+3z^2}{3+z^2}$: it is such that $g(1) = 1$, $g'(1) = 1$ but g is not the identity of Δ.

In 1994 Burns and Krantz [2] have established exact terms in which the rigidity at the boundary can be stated; they proved the following result:

Theorem 1 (Burns-Krantz) *Let Δ be the unit disc centered at $0 \in \mathcal{C}$. Let $f : \Delta \to \Delta$ be a holomorphic map from the unit disc to itself such that*

$$f(\xi) = \xi + O(|\xi - 1|^4)$$

when $\xi \to 1$. Then f is the identity on Δ.

In the same paper the authors generalize their result, first to the case of the open unit ball of $\mathcal{C}^n$, and then to strictly convex and to strongly pseudo-convex domains of $\mathcal{C}^n$. For these classes of domains in several complex variables, the tecnique is to reduce to the one dimensional case; the principal geometric tool is the construction of a family of mappings of the unit disc into the domain such that the image analytic discs (the complex geodesics) are holomorphic retracts of the entire domain. Thanks to the work of Lempert (see [11]), we know that such special analytic discs exist for a large family of smoothly bounded domains.

In this paper we study Cartan-type rigidity at the boundary for holomorphic mappings of the domain $D^2 = \{z \equiv (z_1, z_2) \in \mathcal{C}^2 \ : \ |z_1| + |z_2| < 1\}$, i.e. of the L^1-ball of $\mathcal{C}^2$. This is a non-strictly convex domain, whose boundary is smooth only at the points of $\partial D^2 \cap (\mathcal{C}\setminus\{0\})^2$ where D^2 is strictly pseudo-convex. The domain D^2 came into scene in 1927, when N. Kritikos [10] proved, as one of the first applications of the Carathéodory distance, that every holomorphic automorphism of D^2 leaves the origin fixed; a result which Thullen [18] reobtained four years later, while constructing the group of all holomorphic automorphisms of bounded Reinhardt domains in $\mathcal{C}^2$. Since then, the peculiar complex structure of the domain $D^n = \{(z_1, \ldots, z_n) \in \mathcal{C}^n : |z_1| + \ldots + |z_n| < 1\}$, $n \geq 2$, has been investigated by several authors (see, e.g., [19], [4], [14]) and in particular the family of all complex geodesics of such domains has been described.
The Cartan-type rigidity results proved by Burns-Krantz and Huang [24] for large classes of domains rely upon hypotheses of smoothness and pseudo-convexity which do not hold at all points of ∂D^2. Moreover the techniques used for the unit ball of $\mathcal{C}^n$ cannot be adapted to this situation, because of the non-transitivity of the group of holomorphic automorphisms of D^2. Here we have explicitly studied several families of complex geodesics of D^2 touching given boundary points and have found the corresponding holomorphic retracts in the sense of Lempert [12]. A careful investigation of the holomorphic extendibility of the Lempert-type retracts to the boundary of D^2 and a few results on their "uniqueness" allows the proof of the following:

Theorem 2 *Let D^2 be the domain $\{z \equiv (z_1, z_2) \in \mathcal{C}^2 \ : \ |z_1| + |z_2| < 1\}$. Let $f : D^2 \longrightarrow D^2$ be a holomorphic map from the domain D^2 to itself such that*

$$f(z) = z + O(|\, z - P \,|^4) \quad \textit{when } z \to P \ ,$$

where $P \in \partial D$. *Then*

$$f(z) \equiv z \ \ in \ D^2 \ .$$

As a consequence, analogous rigidity results are obtained for special points of the boundary of D^n, $n > 2$. A full generalization of the above result to the case of $n > 2$ is now under investigation.

The techniques introduced to prove the above results depend on the structure of the family of complex geodesics of the domain involved and could be used, in principle, independently of the geometry of the boundary, for classes of non-regular domains.

2 Preliminaries

If Ω is a domain in $\mathcal{C}^n$, K_Ω and C_Ω will denote, respectly, the Kobayashi and Caratheodory pseudo-distances associated to Ω (see [23]). If Ω is convex, it turns that $K_\Omega \equiv C_\Omega$ (see [11]). Let $g : \Delta \longrightarrow \Omega$ be a holomorphic map. Then g is a *complex geodesics* for C_Ω if, for all $\xi, \eta \in \Delta$,

$$C_\Omega(g(\xi), g(\eta)) = \omega(\xi, \eta) \ ,$$

where ω denotes the Poincarè distance of Δ. Several results have been obtained in the study of existence and uniqueness of complex geodesics joining two given points of a convex domain (see [20],[11]). The complex geodesics of a convex domain Ω have been also characterized as the solutions of an extremum problem (see [11]).

As for their computation, very few results have been obtained, with different techniques. In the case of the L^1 open, unit ball of $\mathcal{C}^n$, $D^n = \{(z_1, \ldots, z_n) \in \mathcal{C}^n : |z_1| + \ldots + |z_n| < 1\}$, complex geodesics are computed in [4]:

Theorem 3 (Gentili) *Let* $g = (g_1, \ldots, g_n) : \Delta \longrightarrow D^n$ *be a complex geodesics of* D^n, *continuous on* $\bar{\Delta}$. *Then there exist:*

(i) K, J, S *non negative integers such that* $K + J + S = n$,

(ii) $\alpha_1, \ldots, \alpha_{K+J} \in \bar{\Delta}$ *not all coinciding with the same* $\alpha \in \partial\Delta$,

(iii) $t_1, \ldots, t_{K+J} \in \mathbb{R}^+_*$,

(iv) $\varphi_1, \ldots, \varphi_{K+J} \in \mathbb{R}$,

such that, up to permutation of the components:

$$\begin{aligned}
g_k(z) &= e^{i\varphi_k} \frac{t_k(z - \alpha_k)(1 - \bar{\alpha_k} z)}{r(1 - \bar{\gamma} z)^2} && for \quad 1 \le k \le K \ , \\
g_j(z) &= e^{i\varphi_j} \frac{t_j(1 - \bar{\alpha_j} z)^2}{r(1 - \bar{\gamma} z)^2} && for \quad K \le j \le K + J \ , \\
g_s(z) &\equiv 0 && for \quad K + J < s \le n \ ,
\end{aligned}$$

where $r \in R_*^+$ *and* $\gamma \in \Delta$ *are such that:*

$$r(z-\gamma)(1-\bar{\gamma}z) = -\left[\sum_{j=1}^{K+J} t_j \bar{\alpha}_j\right] z^2 + \left[\sum_{j=1}^{K+J} t_j(|\alpha_j|^2+1)\right] z - \left[\sum_{j=1}^{K+J} t_j \alpha_j\right] .$$

3 Complex geodesics in D^2 passing through a boundary point

In this section we will study existence of complex geodesics in D^2 passing through a boundary point $P \in \partial D^2$. Since the group of holomorphic automorphisms of D^2 is not transitive on D^2, the possible choices of the point P at boundary of D^2 are not all equivalent. We suppose (up to holomorphic automorphisms) that the boundary point is $P = (1,0)$; notice that the boundary of D^2 is not smooth at P. Let

$$D^2 = \left\{ z \equiv (z_1, z_2) \in \mathcal{C}^2 \ : \ |z_1| + |z_2| < 1 \right\}$$

and let

$$g = (g_1, g_2) : \Delta \longrightarrow D^2$$

be a complex geodesic in D^2, without identically zero components, such that

$$g(1) = (1,0) \in \partial D^2 .$$

Theorem 3 implies the existence of $\alpha_1, \alpha_2 \in \bar{\Delta}$ not both coinciding with the same $\alpha \in \partial\Delta$, of $t_1, t_2 \in \mathbb{R}^+$ and of $\varphi_1, \varphi_2 \in \mathbb{R}$ such that the components of g are in one of the following forms:

$$g_1(\xi) = e^{i\varphi_1}\frac{t_1(\xi-\alpha_1)(1-\bar{\alpha}_1\xi)}{r(1-\bar{\gamma}\xi)^2} , \quad g_2(\xi) = e^{i\varphi_2}\frac{t_2(1-\bar{\alpha}_2\xi)^2}{r(1-\bar{\gamma}\xi)^2} ,$$

or

$$g_1(\xi) = e^{i\varphi_1}\frac{t_1(\xi-\alpha_1)(1-\bar{\alpha}_1\xi)}{r(1-\bar{\gamma}\xi)^2} , \quad g_2(\xi) = e^{i\varphi_2}\frac{t_2(\xi-\alpha_2)(1-\bar{\alpha}_2\xi)}{r(1-\bar{\gamma}\xi)^2} ,$$

or

$$g_1(\xi) = e^{i\varphi_1}\frac{t_1(1-\bar{\alpha}_1\xi)^2}{r(1-\bar{\gamma}\xi)^2} , \qquad g_2(\xi) = e^{i\varphi_2}\frac{t_2(1-\bar{\alpha}_2\xi)^2}{r(1-\bar{\gamma}\xi)^2} ,$$

where $r \in \mathbb{R}^+$ and $\gamma \in \Delta$ satisfy

$$(1) \qquad r(\xi-\gamma)(1-\bar{\gamma}\xi) =$$
$$= -(t_1\bar{\alpha}_1 + t_2\bar{\alpha}_2)\xi^2 + [(1+|\alpha_1|^2)t_1 + (1+|\alpha_2|^2)t_2]\xi - (t_1\alpha_1 + t_2\alpha_2)$$

for all $\xi \in \Delta$. We shall find a family of complex geodesics of D^2 passing through the boundary point $P = (1,0) \in \partial D^2$, and prove that it covers at least an open subset of D^2.

Let us consider all the geodesics of the form:

$$g_1(\xi) = e^{i\varphi_1}\frac{t_1(1-\bar{\alpha}_1\xi)^2}{r(1-\bar{\gamma}\xi)^2}\ , \qquad g_2(\xi) = e^{i\varphi_2}\frac{t_2(1-\bar{\alpha}_2\xi)^2}{r(1-\bar{\gamma}\xi)^2}\ .$$

Condition $g(1) = (1,0)$ yields

$$(2)\qquad \begin{cases} e^{i\varphi_1}\dfrac{t_1(1-\bar{\alpha}_1)^2}{r(1-\bar{\gamma})^2} & = 1\ , \\[2ex] e^{i\varphi_2}\dfrac{t_2(1-\bar{\alpha}_2)^2}{r(1-\bar{\gamma})^2} & = 0\ . \end{cases}$$

Hence $\alpha_2 = 1$. By replacing $\alpha_2 = 1$ in equation (1) we obtain

$$-(r\bar{\gamma})\xi^2 + r(1+|\gamma|^2)\xi - r\gamma = -(t_1\bar{\alpha}_1 + t_2)\xi^2 + [(1+|\alpha_1|^2)t_1 + 2t_2]\xi - (t_1\alpha_1 + t_2)\ .$$

By the identity principle for polynomials and by system (2) we get

$$(3)\qquad \begin{cases} t_1\bar{\alpha}_1 + t_2 & = r\bar{\gamma}\ , \\ (1+|\alpha_1|^2)t_1 + 2t_2 & = r(1+|\gamma|^2)\ , \\ e^{i\varphi_1}t_1(1-\bar{\alpha}_1)^2 & = r(1-\bar{\gamma})^2\ , \end{cases}$$

and the square roots in the last equation are

$$(4)\qquad e^{i\frac{\varphi_1}{2}}\sqrt{t_1}(1-\bar{\alpha}_1) = \sqrt{r}(1-\bar{\gamma})\ ,$$

$$e^{i\frac{\varphi_1}{2}}\sqrt{t_1}(1-\bar{\alpha}_1) = \sqrt{r}(\bar{\gamma}-1)\ .$$

Let us study now equations (3) and (4). We have

$$(5)\qquad \begin{cases} t_1\bar{\alpha}_1 + t_2 & = r\bar{\gamma}\ , \\ (1+|\alpha_1|^2)t_1 + 2t_2 & = r(1+|\gamma|^2)\ , \\ e^{i\frac{\varphi_1}{2}}\sqrt{t_1}(1-\bar{\alpha}_1) & = \sqrt{r}(1-\bar{\gamma})\ . \end{cases}$$

From the first and the third equations we obtain:

$$(6)\qquad \alpha_1 \ = \ \frac{t_2 + \sqrt{t_1 r}\, e^{-i\frac{\varphi_1}{2}} - r}{\sqrt{t_1 r}\, e^{-i\frac{\varphi_1}{2}} - t_1} =: \alpha_0\ ,$$

$$(7)\qquad \gamma \ = \ \frac{(t_1+t_2)\sqrt{t_1 r}\, e^{-i\frac{\varphi_1}{2}} - rt_1}{r(\sqrt{t_1 r}\, e^{-i\frac{\varphi_1}{2}} - t_1)} =: \gamma_0$$

(α_0 and γ_0 satisfy also the second one). The two conditions $|\alpha_0| \le 1$ and $|\gamma_0| < 1$ become

$$(8)\qquad \begin{cases} |\alpha_0|^2 \ = \ \dfrac{(t_2-r)^2 + 2(t_2-r)\sqrt{t_1 r}\cos\frac{\varphi_1}{2} + t_1 r}{t_1(r - 2\sqrt{t_1 r}\cos\frac{\varphi_1}{2})} \le 1\ , \\[3ex] |\gamma_0|^2 \ = \ \dfrac{(t_1+t_2)^2 + rt_1 - 2(t_1+t_2)\sqrt{t_1 r}\cos\frac{\varphi_1}{2}}{r(r - 2\sqrt{t_1 r}\cos\frac{\varphi_1}{2})} < 1\ , \end{cases}$$

$$\begin{cases} t_2^2 - 2t_2(r - \sqrt{t_1 r}\cos\frac{\varphi_1}{2}) + r^2 - t_1^2 + 2(t_1 - r)\sqrt{t_1 r}\cos\frac{\varphi_1}{2} \leq 0 \ , \\ t_2^2 - 2t_2(\sqrt{t_1 r}\cos\frac{\varphi_1}{2} - t_1) - r^2 + t_1^2 - 2(t_1 - r)\sqrt{t_1 r}\cos\frac{\varphi_1}{2} < 0 \ , \end{cases}$$

we find a solution for t_2:

$$r + t_1 - 2\sqrt{t_1 r}\cos\frac{\varphi_1}{2} < t_2 < r - t_1$$

with

$$r > t_1 \quad \text{and} \quad \cos\frac{\varphi_1}{2} > \sqrt{\frac{t_1}{r}} \ .$$

So a family Γ of complex geodesics passing through the point $(1,0)$ is given by

$$g_1(\xi) = e^{i\varphi_1}\frac{t_1(1-\bar{\alpha_0}\xi)^2}{r(1-\bar{\gamma_0}\xi)^2} \ , \tag{9}$$

$$g_2(\xi) = e^{i\varphi_2}\frac{t_2(1-\xi)^2}{r(1-\bar{\gamma_0}\xi)^2} \ , \tag{10}$$

with α_0 and γ_0 as in (7), where $\varphi_1, \varphi_2 \in \mathbb{R}$ and $r, t_1, t_2 \in \mathbb{R}^+_*$ satisfy

$$\begin{cases} t_1 < r \ , \\ r + t_1 - 2\sqrt{t_1 r}\cos\frac{\varphi_1}{2} < t_2 < r - t_1 \ , \\ \cos\frac{\varphi_1}{2} > \sqrt{\frac{t_1}{r}} \ . \end{cases} \tag{11}$$

Remark: $\alpha_0 \neq \gamma_0$, otherwise $g_1(\xi) = e^{i\varphi_1}\dfrac{t_1}{r}$ would be constant.

Let $Q = (z_0, w_0) \in D^2$. Let us find conditions on the point Q so as to guarantee the existence of a complex geodesic $g \in \Gamma$ such that

$$(\, g(1) = (1,0)\,) \quad \text{and} \quad g(0) = (z_0, w_0) \ .$$

We have:

$$g_1(\xi) = e^{i\varphi_1}\frac{t_1(1-\bar{\alpha_0}\xi)^2}{r(1-\bar{\gamma_0}\xi)^2} \ , \tag{12}$$

$$g_2(\xi) = e^{i\varphi_2}\frac{t_2(1-\xi)^2}{r(1-\bar{\gamma_0}\xi)^2} \ , \tag{13}$$

with α_0, γ_0 given by (7). The two conditions $g_1(0) = z_0$ and $g_2(0) = w_0$ yield

$$\begin{cases} e^{i\varphi_1}\dfrac{t_1}{r} = z_0 & \Rightarrow arg(z_0) = \varphi_1 \ , \\ e^{i\varphi_2}\dfrac{t_2}{r} = w_0 & \Rightarrow arg(w_0) = \varphi_2 \ . \end{cases}$$

Let $p = |z_0|$ and $q = |w_0|$; we get

$$\begin{cases} t_1 = rp \\ t_2 = rq \end{cases} .$$

The values of t_1, t_2 and φ_1 satisfy conditions (11). Hence

$$\begin{array}{ll} 1) & rp < r \text{ holds because } 0 \leq p < 1 \ , \\ 2) & \cos\frac{\varphi_1}{2} > \sqrt{p} \quad \Rightarrow -2\arccos\sqrt{p} < \varphi_1 < 2\arccos\sqrt{p} \ , \\ 3) & p + 1 - 2\sqrt{p}\cos\frac{\varphi_1}{2} < s < 1 - p \ . \end{array}$$

So we will have a complex geodesic $g \in \Gamma$ passing through Q if we choose:

$$\begin{array}{ll} \varphi_1 = arg(z_0) \ , & t_1 = r|z_0| \ , \\ \varphi_2 = arg(w_0) \ , & t_2 = r|w_0| \ , \end{array}$$

with

$$1 + |z_0| - 2\sqrt{|z_0|}\, cos\left(\frac{arg(z_0)}{2}\right) < |w_0| < 1 - |z_0| \ , \tag{14}$$

$$-2\arccos\sqrt{|z_0|} < arg(z_0) < 2\arccos\sqrt{|z_0|} \ . \tag{15}$$

It turns out that the set

$$S = \{(z_0, w_0) \in D^2 \ : \ (z_0, w_0) \text{ satisfy (14),(15)}\}$$

is open on D^2 and that the complex geodesics passing through the point $(1,0)$ do not intersect the axes twice.

For any complex geodesics passing through the boundary point $(1,0) \in \partial D^2$ we shall find a "inverse" function in the sense of the following:

Proposition 1 *Let $g = (g_1, g_2) : \Delta \longrightarrow D^2$ be a complex geodesic such that*

$$g(1) = (1,0) \in \partial D^2 \quad , \quad g(0) = (z_0, w_0) \in D^2 \ .$$

Then, there exists a holomorphic function $F_g : D^2 \longrightarrow \Delta$ such that $F_g \circ g = id_\Delta$.

Proof.

The complex geodesic g maps $\partial\Delta$ into ∂D^2 and it can be continuosly extended to $\bar{\Delta}$; as we noticed $g_1 \neq 0$ on $\partial\Delta$ and $g_2(\xi) = 0 \Leftrightarrow \xi = 1$. Moreover, the unit normal vector $\nu(z)$ to ∂D^2 in $z = (z_1, z_2) \in \partial D^2$ is given by

$$\nu(z) = \left(\frac{z_1}{|z_1|}, \frac{z_2}{|z_2|}\right) \ .$$

Hence

$$\nu\left(g(e^{i\theta})\right) = \left(\frac{g_1(e^{i\theta})}{|g_1(e^{i\theta})|}, \frac{g_2(e^{i\theta})}{|g_2(e^{i\theta})|}\right) \ ,$$

with $\theta \in [0, 2\pi]$. By a result due to Lempert (see[12]), there exsists a positive real function

$$\tilde{r} : \partial\Delta \longrightarrow \mathbb{R}^+$$

such that the mapping

$$\tilde{G} : \partial\Delta \longrightarrow \mathcal{C}^2 \ .$$

defined by

$$\xi \longmapsto \xi \tilde{r}(\xi)\overline{\nu\left(g(\xi)\right))}$$

extends to a holomorphic mapping

$$G = (G_1, G_2) : \Delta \longrightarrow \mathcal{C}^2 \ .$$

To see that g has a holomorphic "left-inverse", we consider the equation

$$< z - g(\xi)\ ,\ G(\xi) >= 0 \tag{16}$$

for the unknown $\xi \in \Delta$, z being a fixed point in D^2. Here $< a, b >= \sum_{k=1}^{2} a_k b_k$ for $a = (a_i)_{i=1,2}, b = (b_j)_{j=1,2} \in \mathcal{C}^2$.

For each $z \in D^2$, equation (16) has exactly one solution $\xi \in \Delta$. Indeed, the number of the solutions equals $wind\,\Psi$, the winding number of the function

$$\Psi(\xi) =< z - g(\xi),\ G(\xi) >$$

on $\partial\Delta$ (see[1]). Now

$$wind\Psi(\xi) = \underbrace{wind\xi}_{=1} + \underbrace{wind < z - g(\xi), \tilde{r}(\xi)\overline{\nu(g(\xi))} >}_{=0} \tag{17}$$

since

$$Re < z - g(\xi), \tilde{r}(\xi)\overline{\nu(g(\xi))} >< 0 \quad , \quad \forall \xi \in \Delta$$

by convexity. Thus the second summand in (17) is zero. Therefore, we have a function

$$F_g : D^2 \longrightarrow \Delta$$

which assigns to each $z \in \Delta$ the corresponding solution $\xi \in \Delta$ of (16); F_g is holomorphic and

$$F_g \circ g = id_\Delta \ .$$

We assert that $F_g : D^2 \longrightarrow \Delta$ is holomorphic in a neighborhood of $P = (1, 0) \in \partial D^2$. Indeed, the components g_1 and g_2 of the complex geodesic g are continous and holomorphic in a neighborhood of Δ; moreover(see[4]), we have

$g_1(\xi)G_1(\xi) = t_1(\xi - \alpha_0)(1 - \bar{\alpha_0}\xi)$,

$g_2(\xi)G_2(\xi) = -t_2(1 - \xi)^2$.

Then

$$\begin{aligned} G_1(\xi) &= re^{-i\varphi_1}\frac{(\xi - \alpha_0)(1 - \bar{\gamma_0}\xi)^2}{(1 - \bar{\alpha_0}\xi)} \ , \\ G_2(\xi) &= -re^{-i\varphi_2}(1 - \bar{\gamma_0}\xi)^2 \ . \end{aligned}$$

Since $\alpha_0 \neq 1$, G_1 is holomorphic on $\Delta \cup V$, where V is a neighborhood of $1 \in \partial\Delta$, while G_2 is holomorphic in a neighborhood of Δ.

Let us consider now the mapping defined in (16):

$$\Phi(z_1, z_2, \xi) =:< z - g(\xi), G(\xi) > \ .$$

The mapping Φ is holomorphic on $\tilde{V} \times \Delta \cup V$ where $\tilde{V}$ is a neighborhood of D^2. Let

$$\Theta = \{\Phi(z_1, z_2, \xi) = 0\} \ .$$

We have $(1,0,1) \in \Theta$ and $\dfrac{\partial}{\partial\xi}\Phi(1,0,1) = -g_1'(1)G_1(1) \neq 0.$

Therefore, by the implicit function theorem (see [3]), ξ can be expressed as a holomorphic function of (z_1, z_2) on D^2 and on a neighborhood of the point $P = (1,0) \in \partial D^2$.

4 A rigidity result for D^2

Let us now prove a rigidity result for the domain $D^2 \subset \mathcal{C}^2$.

Theorem 4 *Let D^2 be the domain $\{z \equiv (z_1, z_2) \in \mathcal{C}^2 \ : \ |z_1| + |z_2| < 1\}$. Let $f : D^2 \longrightarrow D^2$ be a holomorphic map from the domain D^2 to itself such that*

$$f(z) = z + O(| z - P |^4) \quad when \ z \to P \ ,$$

where $P \in \partial D$. Then

$$f(z) \equiv z \ in \ D^2 \ .$$

Proof.

We will consider first the case in which $P \equiv (1,0)$. Let $(z_0, w_0) \in D^2$ be a given point such that (14) and (15) hold. Let $g = (g_1, g_2) : \Delta \longrightarrow D^2$ be a complex geodesic on D^2 such that

$$g(1) = (1,0) \quad \text{and} \quad g(0) = (z_0, w_0) \ .$$

Its components are given by

$$g_1(\xi) = e^{i\varphi_1}\frac{t_1(1-\bar{\alpha}_0\xi)^2}{r(1-\bar{\gamma}_0\xi)^2} \ , \tag{18}$$

$$g_2(\xi) = e^{i\varphi_2}\frac{t_2(1-\xi)^2}{r(1-\bar{\gamma}_0\xi)^2} \ , \tag{19}$$

with α_0 and γ_0 as in (7), and where $\varphi_1, \varphi_2 \in \mathbb{R}$ and $r, t_1, t_2 \in \mathbb{R}_*^+$ satisfy (11).

Let $F_g : D^2 \longrightarrow \Delta$ be the "inverse" holomorphic mapping of g defined in proposition (1). Then the function $H =: F_g \circ f \circ g : \Delta \longrightarrow \Delta$,

$$\begin{array}{ccccccc} \Delta & \xrightarrow{g} & D^2 & \xrightarrow{f} & D^2 & \xrightarrow{F_g} & \Delta \\ 1 & \longmapsto & (1,0) & \longmapsto & (1,0) & \longmapsto & 1 \end{array}$$

is well-defined. Moreover, H satisfies the hypotheses of Burns-Krantz's Theorem 1. Indeed, by considering the function

$$\psi(t) = tf(g(\xi)) + (1-t)g(\xi)$$

we have

$$F_g(f(g(\xi))) - \xi = F_g(f(g\xi))) - F_g(g(\xi)) = F_g(\psi(1)) - F_g(\psi(0)) \ .$$

Thus, we get

$$|F_g(\psi(1)) - F_g(\psi(0))| \leq \left|dF_g\left(\psi(\tilde{t})\right)\right| |f(g(\xi)) - g(\xi)| \ .$$

If $M \geq \max_K\{\|dF_g\|\}$, where K is a compact neighborhood of $P \in \partial D^2$ on which F_g is holomorphic, we obtain

$$\frac{|F_g(f(g(\xi))) - F_g(g(\xi))|}{|\xi - P|^4} \leq M \ \frac{|f(g(\xi)) - g(\xi)|}{|\xi - P|^4} = L(\xi) \ .$$

For $\xi \to P$, $L(\xi)$ is bounded and non zero. Hence

$$F_g(f(g(\xi))) - \xi = O(|\xi - P|^4) \ .$$

Then

$$H(\xi) \equiv \xi \quad , \quad \forall \xi \in \Delta \ ,$$

i.e.

$$F_g \circ f \circ g(\xi) = \xi \quad , \quad \forall \xi \in \Delta.$$

In particular:
$(f \circ g) : \Delta \longrightarrow D^2$ is a complex geodesic on D^2 passing through $(1,0)$: in fact we have $f \circ g(1) = (1,0)$ and for $\xi_0, \xi_1 \in \Delta$ we get

$$\begin{aligned}\omega(\xi_0,\xi_1) \ &= \omega(F_g \circ f \circ g(\xi_0) \ , \ F_g \circ f \circ g(\xi_1)) \leq \\ &\leq C_D(f \circ g(\xi_0) \ , \ f \circ g(\xi_1)) \leq C_D(g(\xi_0) \ , \ g(\xi_1)) = \\ &= \omega(\xi_0,\xi_1) \ .\end{aligned}$$

It follows

$$C_D(f \circ g(\xi_0) \ , \ f \circ g(\xi_1)) = \omega(\xi_0,\xi_1) \ ,$$

and the assertion is proved. The second component of $f \circ g$, according to theorem (3), is given by

$$(f \circ g)_2 = \tilde{a_2} \, \frac{(1-\xi)^2}{(1-\tilde{\gamma}\xi)^2}$$

with $\tilde{a_2} \in \mathcal{C}$ and $\tilde{\gamma} \in \Delta$. The "inverse" function F_g of g also "inverts" $f \circ g$, because $\mathrm{F}_g(f \circ g) = id_\Delta$. Hence

$$< (f \circ g)(\xi) - g(\xi) \ , \ G(\xi) >= 0 \quad , \quad \forall \xi \in \Delta. \tag{20}$$

$f \circ g \equiv g$ for $\xi \to P$ up to the third order. Indeed

$$\lim_{\xi \to 1} \frac{|f(g(\xi)) - g(\xi)|}{|\xi - P|^3} = \lim_{\xi \to 1} \frac{|f(g(\xi)) - g(\xi)|}{|g(\xi) - P|^3} \frac{|g(\xi) - P|^3}{|\xi - P|^3} = 0 .$$

Let us compute the values of the derivative of g at $\xi = 1$:

$$\begin{aligned} g_2'(1) &= 0 , \\ g_2''(1) &= \frac{2a_2}{(1-\bar{\gamma}_0)^2} , \\ g_2'''(1) &= \frac{12 a_2 \bar{\gamma}_0}{(1-\bar{\gamma}_0)^3} , \end{aligned}$$

where a_2 is the coefficient appearing in (19). Since $f \circ g \equiv g$ for $\xi \to P$ up to the third order, we obtain

$$\begin{cases} \dfrac{2a_2}{(1-\bar{\gamma}_0)^2} = \dfrac{2\tilde{a}_2}{(1-\bar{\tilde{\gamma}})^2} , \\[2ex] \dfrac{12 a_2 \bar{\gamma}_0}{(1-\bar{\gamma}_0)^3} = \dfrac{12 \tilde{a}_2 \bar{\tilde{\gamma}}}{(1-\bar{\tilde{\gamma}})^3} . \end{cases}$$

By combining the two equations, we get

$$\gamma_0 = \tilde{\gamma} ,$$

which directly implies that $a_2 = \tilde{a}_2$; hence

$$(f \circ g)_2 \equiv g_2 . \tag{21}$$

It follows that

$$|(f \circ g)_1(\xi)| = |g_1(\xi)| = 1 - |g_2(\xi)| \quad , \quad \forall \xi \in \partial\Delta .$$

Therefore, for all $\xi \in \partial\Delta$, we have

$$(f \circ g)_1(\xi) = e^{i\vartheta(\xi)} g_1(\xi) . \tag{22}$$

Let us consider the components of (20): for all $\xi \in \partial\Delta$ we have

$$< \left(e^{i\vartheta(\xi)} g_1(\xi), g_2(\xi)\right) - (g_1(\xi), g_2(\xi)) \, , \, \xi \tilde{r}(\xi) \left(\overline{\frac{g_1(\xi)}{|g_1(\xi)|}}, \overline{\frac{g_2(\xi)}{|g_2(\xi)|}} \right) >= 0 .$$

Using (21) and (22), we get

$$\left(e^{i\vartheta(\xi)} - 1\right) \xi \tilde{r}(\xi) |g_1(\xi)| = 0 .$$

Since $\tilde{r} \neq 0$ (except for a set of measure zero), and $|g_1(\xi)| \neq 0 \quad \forall \xi \in \partial\Delta$, we obtain

$$e^{i\vartheta(\xi)} = 1 \qquad \text{a.e. on } \partial\Delta ,$$

hence

$$g_1(\xi) \equiv (f \circ g)_1(\xi) \quad \text{a.e. on } \partial\Delta.$$

Therefore, we conclude that

$$(f \circ g)(\xi) \equiv g(\xi) \quad , \quad \forall \xi \in \Delta.$$

In particular, when $\xi = 0$, we have:

$$f(z_0, w_0) = (z_0, w_0) \ .$$

Then f is the identity map on the set A of points satisfying (14) and (15); since A is an open set of D^2, so we conclude

$$f(z) \equiv z \quad \forall z \in D^2.$$

Remark that the result obtained still holds if the point $P \equiv (1,0)$ is replaced with the point $Q \equiv (e^{i\vartheta}, 0)$ or $Q \equiv (0, e^{i\vartheta})$ $(\forall \vartheta \in R)$. In fact in both cases there exists a holomorphic automorphism of D^2 which trasforms the point Q into P. When P belongs to the " smooth" boundary of D^2 (i.e. when both the coordinates have moduli strictly smaller than 1) the same results hold, because in a neighborhood of P the domain is strictly pseudo-convex and the tecniques developed by Burns and Krantz in [2] can be directly applied to obtain the assertion.

We shall extend theorem (4) to the case of "special" points of the boundary of $D^n \subset \mathcal{C}^n$:

Corollary 1 *Let $D^n = \{(z_1, \ldots, z_n) \in \mathcal{C}^n \ : \ |z_1| + \ldots + |z_n| < 1\}$. Let $f : D^n \longrightarrow D^n$ be a holomorphic map from the domain D^n to itself such that*

$$f(z) = z + O(| \ z - P \ |^4) \quad for \ \ z \to P \ ,$$

where $P \in \partial D^n \cap (\mathcal{C} \backslash \{0\})^n$ or $P \in \partial D^n \cap \partial(\Delta \times \ldots \times \Delta)$. Then

$$f(z) \equiv z \ \ on \ D^n \ .$$

Proof.

We will consider first the case in which $P \equiv (1, 0, \ldots, 0)$. Let $g = (g_1, \ldots, g_n) : \Delta \longrightarrow D^n$ be a complex geodesic on D^n, whose components are given in Theorem 3. Since $g(1) = P$, we have

$$\alpha_1 \in \bar{\Delta} \setminus \{1\} \quad , \quad \alpha_2 = \alpha_3 = \ldots = \alpha_n = 1 \ .$$

We obtain a system as in (5) :

$$\begin{cases} r\bar{\gamma} & = t_1\bar{\alpha_1} + t_2 + \ldots + t_n \ , \\ r(1 + |\gamma|^2) & = t_1(1 + |\alpha_1|^2) + 2t_2 + \ldots + 2t_n \ , \\ r(1 - \bar{\gamma})^2 & = e^{i\varphi_1} t_1 (1 - \bar{\alpha_1})^2 \ . \end{cases}$$

By denoting $\tilde{t_2} =: t_2 + \ldots + t_n$, we find the same solutions α_0 and γ_0 as in (7) with t_2 replaced by $\tilde{t_2}$; we also obtain similar conditions to (11) for $\varphi_j \in R$ and $t_j \in R^+_*$ $(j = 1, \ldots, n)$. Analogous arguments to those used in the proof of Theorem 4 can be used to prove the assertion.

Remark that the statement still holds if the point $P = (1, 0, \ldots, 0)$ is replaced with the point $Q = (0, \ldots, e^{i\theta}_j, 0, \ldots, 0) \in \partial D^n$ (with $j = 1, \ldots n$, and $\theta \in R$); in fact there exists a holomorphic automorphism of D^n, which trasforms the point Q into P.

When $P \in \partial D^n$ belongs to the "smooth" boundary of D^n, (i.e. when all the coordinates of $P \in D^n$ have moduli strictly greater than 0), the assertion holds, because in a neighborhood of P the domain is strictly pseudo-convex and the techniques developed by Burns and Krantz in [2] can be directly applied to obtain the assertion.

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