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# An example of a differential calculus on the quantum complex n-space*

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**Abstract.** Modified notions of derivation and differential form on the noncommutative space $\mathbb{C}_q^n$ are introduced. It is shown that in this way a first order differential calculus in the sense of Woronowicz is obtained.

## Introduction

Recently there is a growing number of articles concerning the theory of so-called "quantum groups". For a survey and general references see the article [9] in this volume. As mentioned there one considers not only quantum groups but also other noncommutative spaces which are comodules of a quantum group. $\mathbb{C}_q^n$ is the most simple example of such an object.

It is an important task to introduce analogues of notions of differential geometry on such noncommutative spaces ([2]). The most fundamental notions of this kind are those of tangent vectors and differential forms, whose definition will be our concern in this note. Since all the information about a noncommutative space is encoded in an algebra one has to define these notions in purely algebraic terms. By an algebra we always mean an associative algebra over the complex numbers $\mathbb{C}$ with unity $I$. The symbols $\mathbb{Z}$ and $\mathbb{N}$ denote the integers and the nonnegative integers respectively. Throughout the paper we use the so-called Einstein convention: We always form sums over pairs of equal upper and lower indices.

On a classsical space (differentiable manifold) one introduces tangent vectors as equivalence classes of curves starting from a point and defines differential forms as duals to vector fields. From this geometric point of view, everything is based on the notion of a vector. For commutative algebras $A$ (with the example $A = C^\infty(M)$, $M$ a manifold, in mind) it is quite clear how to translate this approach into an algebraic language: One starts with $\mathrm{Der}(A)$, the Lie algebra of derivations of the algebra, and defines $\bigwedge^1(A)$, the space of 1-forms on $A$, as the space of $A$-linear maps $Der(A) \longrightarrow A$. This is possible because $Der(A)$ is an $A$-(bi)module for commutative $A$. For noncommutative $A$, $\mathrm{Der}(A)$ isn't a left or right $A$-module in a natural way, and one could define $\bigwedge^1(A)$ only as the space of $\mathbb{C}$-linear maps $\mathrm{Der}(A) \longrightarrow A$ which is much to big in the classical case (contains even nonlocal mappings). One can try to replace the $A$-linearity condition by another one using e.g. certain maximal ideals of $A$ as analogues of points (see [5]). However, this defect may be taken equally well as a hint to replace the notion of derivation by something else.

There is another approach to differential forms on a general associative algebra which, at first sight, $A$ does not use derivations ([12]):

**Definition 1.** A pair $(\Gamma, d)$ is called *first order differential calculus* on $A$ if

1. $\Gamma$ is an $A$-bimodule.
2. $d : A \longrightarrow \Gamma$ is $\mathbb{C}$-linear and satisfies
   (i) $d(ab) = d(a)b + ad(b)$,
   (ii) Any $\alpha \in \Gamma$ can be written $\alpha = \sum_{k=1}^{N} a_k db_k$ with $a_k, b_k \in A, k = 1, \ldots, N$.

Obviously, the classical example $A = C^\infty(M)(M$ a manifold), $\Gamma = \sec(T^*M)$, $d$ the standard differential, is of this type. (Condition 2.(ii) is a mild restriction. It is satisfied at least for manifolds which can be covered by a finite number of charts.)

Now, let us assume that $(\Gamma, d)$ is a given first order differential calculus on $A$, and let $\Gamma$ be in particular a free $A$-bimodule with a finite basis $(\omega^i)_{i=1,\ldots,n}$. This means that any $\alpha \in \Gamma$ can be written $\alpha = a_i\omega^i = \omega^i b_i$ (summation over pairs of equal upper and lower indices, see remark at the end of the second paragraph) with uniquely determined $a_i$ and $b_i \in A$. Then we must have

$$a\omega^i = \omega^j C^i_j(a) \tag{1}$$

with $C^i_j \in L(A) \stackrel{\text{def}}{=} \mathrm{End}_{\mathbb{C}}(A)$,

$$da = \omega^i X^r_i(a) \tag{2}$$

with $X^r_i \in L(A)$. It follows immediately that

$$C^i_j(ab) = C^k_j(a)C^i_k(b) \tag{3}$$

and

$$X^r_i(ab) = X^r_i(a)b + C^j_i(a)X^r_j(b) \tag{4}$$

(Notice that we could obtain similar relations starting from $\omega^i a = B^i_j(a)\omega^j$ and $da = X^l_i(a)\omega^i$.)

The examples of noncommutative algebras appearing in the theory of quantum groups are typically generated by elements $x^i$ (subject to quadratic relations) which may be interpreted as analogues of the classical coordinate functions. In a differential calculus $(\Gamma, d)$ one would like to have the $dx^i$ as a basis of $\Gamma$. Then the above $X^r_i$ should be interpreted as analogues of the partial derivatives $\partial_i$, and formula (4) says that the "partial derivatives" $X^r_i$ do not satisfy the usual Leibniz rule, i.e. they aren't derivations. This is a typical phenomenon for quantum groups (cf. [12]).

We will now present a special example of a differential calculus on $\mathbb{C}^n_q$ starting from a generalized notion of derivation. These q-derivations have properties analogous to all properties of usual derivations. In particular, they form a left $\mathbb{C}^n_q$-module which makes it possible to define differential forms in the classical spirit. In this way we obtain a differential calculus in the sense of Definition 1.

## The algebra $\mathbb{C}_q^n$

We define $\mathbb{C}_q^n$, $q \in \mathbb{C} \setminus \{0\}$ as the quotient algebra

$$\mathbb{C}\langle x^1, \ldots, x^n \rangle / I_R$$

where $\mathbb{C}\langle x^1, \ldots, x^n \rangle$ is the free associative algebra with unity generated by the elements $x^1, \ldots, x^n$, and $I_R$ is the twosided ideal generated by the relations

$$x^i x^j = q x^j x^i \quad , \quad i < j \tag{5}$$

(see [8]). The elements $(x^{1^{i_1}} \cdots x^{n^{i_n}})_{(i_1,\ldots,i_n)\in\mathbb{N}^n}$ form a basis (as a vector space) of $\mathbb{C}_q^n$. We will consider $\mathbb{C}_q^n$ as an $\mathbb{N}^n$-graded algebra with homogeneous components $\mathbb{C}_q^{n(i_1,\ldots,i_n)} = \{\lambda x^{1^{i_1}} \cdots x^{n^{i_n}} \mid \lambda \in \mathbb{C}\}$. Further, $\mathbb{C}_q^{nh}$ denotes the set of homogeneous elements of $\mathbb{C}_q^n$. For homogeneous (basis) elements $x = x^{1^{i_1}} \cdots x^{n^{i_n}}$, $y = x^{1^{k_1}} \cdots x^{n^{k_n}}$ with degrees $g(x) = (i_1, \ldots, i_n)$, $g(y) = (k_1, \ldots, k_n)$ one immediately obtains the following commutation rule

$$xy = q^{m(g(x),g(y))} yx \tag{6}$$

(7)
$$m(g(x), g(y)) = (i_1+\ldots+i_{n-1})k_n+(i_1+\ldots+i_{n-2}-i_n)k_{n-1}+\ldots+(-i_2-\ldots-i_n)k_1$$

The map $m : \mathbb{N}^n \times \mathbb{N}^n \longrightarrow \mathbb{Z}^n$ has two important properties:
1. $m$ is additive in both arguments.
2. $m$ is antisymmetric.

We will even consider $\mathbb{C}_q^n$ as a $\mathbb{Z}^n$-graded algebra setting $\mathbb{C}_q^{n(i_1,\ldots,i_n)} = 0$ if $i_k < 0$ for some $k \in \{1, \ldots, n\}$. Obviously, $m$ can be extended to a mapping $\mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$ defined by (7).

## q-derivations

We will follow the idea that any object related to the algebra $\mathbb{C}_q^n$ should also be $\mathbb{Z}^n$-graded and that a commutation of any two homogeneous objects $a$, $b$ in algebraic manipulations should yield a factor according to the rule $ab = q^{m(g(a),g(b))} ba$ as in the algebra $\mathbb{C}_q^n$. This is a generalization of the well known case of $\mathbb{Z}_2$-graded algebras ([1, 3, 4]), which corresponds to $q = -1$ in our case. (The relations $x^{i^2} = 0$ are missing here.)

**Definition 2.** An element $\partial \in L(\mathbb{C}_q^n)$ is called *homogeneous q-derivation of degree* $g(\partial) \in \mathbb{Z}^n$ if

1. $g(\partial(x)) = g(\partial) + g(x)$,
2. $\partial(xy) = \partial(x)y + q^{m(g(\partial),g(x))} x\partial(y)$

for $x \in \mathbb{C}_q^{nh}$. A q-derivation is a finite sum of homogeneous q-derivations. Also, $\mathrm{Der}_q^h(\mathbb{C}_q^n)$ denotes the set of homogeneous q-derivations, $\mathrm{Der}_q(\mathbf{C}_q^n)$ the vector space of all q-derivations.

Notice that the definition is made in such a way that every q-derivation is a finite sum of homogeneous components. The same remark applies to the definition of q-differential forms to be given later. Therefore, all propositions in this paper need a proof only for homogeneous elements.

**Proposition 1.** *Let $X_1$, $X_2 \in \mathrm{Der}_q^h(\mathbb{C}_q^n)$. Then we have*

$$[X_1, X_2]_q \stackrel{\mathrm{def}}{=} X_1X_2 - q^{m(g(X_1),g(X_2))}X_2X_1 \in \mathrm{Der}_q^h(\mathbb{C}_q^n)$$

*with $g([X_1, X_2]_q) = g(X_1) + g(X_2)$.*

**Proof.** For $x \in \mathbb{C}_q^{nh}, y \in \mathbb{C}_q^n$ we have

$$\begin{aligned}
&(X_1X_2 - q^{m(g(X_1),g(X_2))}X_2X_1)(xy) = X_1(X_2(x)y + q^{m(g(X_2),g(x))}xX_2(y)) \\
&- q^{m(g(X_1),g(X_2))}X_2(X_1(x)y + q^{m(g(X_1),g(x))}xX_1(y)) \\
&= X_1X_2(x)y + q^{m(g(X_1),g(X_2)+g(x))}X_2(x)X_1(y) + q^{m(g(X_2),g(x))}(X_1(x)X_2(y) \\
&+ q^{m(g(X_1),g(x))}xX_1X_2(y)) - q^{m(g(X_1),g(X_2))}(X_2X_1(x)y \\
&+ q^{m(g(X_2),g(X_1)+g(x))}X_1(x)X_2(y) + q^{m(g(X_1),g(x))}(X_2(x)X_1(y) \\
&+ q^{m(g(X_2),g(x))}xX_2X_1(y))).
\end{aligned}$$

The coefficient of the term $X_2(x)X_1(y)$ is

$$q^{m(g(X_1),g(X_2)+g(x))} - q^{m(g(X_1),g(X_2))+m(g(X_1),g(x))} = 0,$$

because $m$ is additive. In the same way the coefficient of $X_1(x)X_2(y)$ is zero. The remaining terms are

$$\begin{aligned}
&(X_1X_2 - q^{m(g(X_1),g(X_2))}X_2X_1)(x)y \\
&+ q^{m(g(X_1)+g(X_2),g(x))}x(X_1X_2(y) - q^{m(g(X_1),g(X_2))}X_2X_1(y)),
\end{aligned}$$

which proves the proposition. ∎

The binary operation $[\cdot,\cdot]_q$ is called q-commutator. In this terminology, $\mathbb{C}_q^n$ itself is q-commutative. The proofs of the following propositions are as simple and direct as the preceding one and make use of the two properties of $m$ only. We therefore leave them to the reader.

**Proposition 2.** *For $X, X_1, X_2, X_3 \in \mathrm{Der}_q^h(\mathbb{C}_q^n)$ we have*

1. $[X, X]_q = 0$.
2. $[X_2, X_1]_q = -q^{m(g(X_2),g(X_1))}[X_1, X_2]_q$.
3. $q^{m(g(X_3),g(X_1))}[[X_1, X_2]_q, X_3]_q + q^{m(g(X_1),g(X_2))}[[X_2, X_3]_q, X_1]_q$ $+ q^{m(g(X_2),g(X_3))}[[X_3, X_1]_q, X_2]_q = 0$. ∎

These are modifications of the usual properties of a Lie algebra. Thus, $\mathrm{Der}_q(\mathbb{C}_q^n)$ could be called a q-Lie algebra.

**Proposition 3.** *For* $a \in \mathbb{C}_q^{n\,h}, X, X_1, X_2 \in \mathrm{Der}_q^h(\mathbb{C}_q^n)$ *we have*

1. $aX \in \mathrm{Der}_q^h(\mathbb{C}_q^n)$ *with* $g(aX) = g(a) + g(X)$.
2. $[aX_1, X_2]_q = a[X_1, X_2]_q - q^{m(g(X_1)+g(a), g(X_2))} X_2(a) X_1$. ■

Condition 1. says that $\mathrm{Der}_q(\mathbb{C}_q^n)$ is a left $\mathbb{C}_q^n$-module. Condition 2. is a modification of well known properties of vector fields.

We will now define analogues of partial derivatives and prove that they form a basis of the left $\mathbb{C}_q^n$-module $\mathrm{Der}_q(\mathbb{C}_q^n)$. First let us notice that derivations (in the usual sense) $\partial_i$ of $\mathbb{C}_q^n$ with the property $\partial_i(x^j) = \delta_i^j I$ do not exist: For example for $i < j$ would follow $\partial_i(x^i x^j) = x^j = \partial_i(q x^j x^i) = q x^j$ (no summation over i!). We will define $\partial_i$ first as operators on $\mathbb{C}\langle x^1, \ldots, x^n\rangle$ by

$$\partial_i(x^j) = \delta_i^j I, \qquad \partial_i(I) = 0,$$
$$\partial_i(x^j x) = \delta_i^j x + q^{m(g(\partial_i), g(x^j))} x^j \partial_i(x)$$

with $g(\partial_i) = -g(x^i) = (0, \ldots, 0, -1, 0, \ldots, 0)$ (-1 at $i$-th position). Here, a homogeneous element of $\mathbb{C}\langle x^1, \ldots, x^n\rangle$ is any product of the $x^1, \ldots, x^n$ times a scalar, and the $j$-th component of the degree of such an element is the number of factors $x^j$ appearing in this element.

**Proposition 4.** *The following statements hold:*

1. $\partial_i(xy) = \partial_i(x)y + q^{m(g(\partial_i), g(x))} x \partial_i(y)$ *for homogeneous* $x$, *i .e.,* $\partial_i$ *is a q-derivation on* $\mathbb{C}\langle x^1, \ldots, x^n\rangle$*).*
2. $\partial_i(I_R) \subset I_R$, *where* $I_R$ *is the ideal defined by the relations* (5). ■

This means that $\partial_i$ project to q-derivations of $\mathbb{C}_q^n$.

**Theorem 1.** *Let* $X \in \mathrm{Der}_q(\mathbb{C}_q^n)$ *(not necessarily homogeneous). Then we have*

$$X = X(x^1)\partial_1 + \ldots + X(x^n)\partial_n. \quad ■$$

The proof can be performed by a direct computation. It is easy to show that the $\partial_i$ are linearly independent as elements of the left $\mathbb{C}_q^n$-module $\mathrm{Der}_q(\mathbb{C}_q^n)$:

$$a^i \partial_i = 0 \Rightarrow a^i \partial_i(x^j) = a^j I = 0 \Rightarrow a^j = 0.$$

Thus, $\mathrm{Der}_q(\mathbb{C}_q^n)$ is a free left $\mathbb{C}_q^n$-module with basis $(\partial_i)_{i=1,\ldots,n}$.

## q-differential forms

**Definition 3.** We denote by $\bigwedge_{0q}^{1\,h}(\mathbb{C}_q^n)$ the set of $\mathbb{C}$- linear maps

$$\alpha\colon \mathrm{Der}_q(\mathbb{C}_q^n) \to \mathbb{C}_q^n$$

with

1. $X \in \mathrm{Der}_q^h(\mathbb{C}_q^n) \Rightarrow \alpha(X) \in \mathbb{C}_q^{nh}$, $g(\alpha(X)) = g(\alpha) + g(X)$. (This defines $g(\alpha) \in \mathbb{Z}^n$.)
2. $\alpha(aX) = q^{m(g(\alpha),g(a))} a\alpha(X)$ for $a \in \mathbb{C}_q^{nh}$.

Further, $\bigwedge_{0q}^1(\mathbb{C}_q^n)$ is the vector space of finite sums of elements of $\bigwedge_{0q}^{1h}(\mathbb{C}_q^n)$. Elements of $\bigwedge_{0q}^1(\mathbb{C}_q^n)$ and $\bigwedge_{0q}^{1h}(\mathbb{C}_q^n)$ are called *q-differential 1-forms* and *homogeneous q-differential 1-forms*, respectively. ∎

Condition 2. replaces $A$-linearity which for commutative algebras $A$ gives a locality condition. Obviously, $\bigwedge_{0q}^1(\mathbb{C}_q^n)$ becomes a $\mathbb{C}_q^n$-bimodule with

$$a\alpha = q^{m(g(a),g(\alpha))}\alpha a, \quad \alpha a = q^{m(g(\alpha),g(a))} a\alpha$$

for $a \in \mathbb{C}_q^{nh}$, $\alpha \in \bigwedge_{0q}^{1h}(\mathbb{C}_q^n)$.

**Definition 4.** We define $d\colon \mathbb{C}_q^n \to \bigwedge_{0q}^1(\mathbb{C}_q^n)$ by

$$da(X) = q^{m(g(a),g(X))} X(a)$$

for $a \in \mathbb{C}_q^n$, $X \in \mathrm{Der}_q^h(\mathbb{C}_q^n)$. ∎

Obviously, we have $g(da) = g(a)$ and

$$da(bX) = q^{m(g(a),g(b)+g(X))} bX(a) = q^{m(g(a),g(b))} bda(X),$$

i. e., $d$ indeed has its values in $\bigwedge_{0q}^1(\mathbb{C}_q^n)$.

**Theorem 2.** *$(\bigwedge_{0q}^1(\mathbb{C}_q^n), d)$ is a first order differential calculus on $\mathbb{C}_q^n$ in the sense of* Definition 1.

**Proof.** Firstly, we know that $\bigwedge_{0q}^1(\mathbb{C}_q^n)$ is a $\mathbb{C}_q^n$-bimodule. Secondly, we compute

$$\begin{aligned}
d(ab)(X) &= q^{m(g(a)+g(b),g(X))} X(ab) \\
&= q^{m(g(a)+g(b),g(X))}(X(a)b + q^{m(g(X),g(a))} aX(b)) \\
&= q^{m(g(a)+g(b),g(X))} X(a)b + q^{m(g(b),g(X))} aX(b). \\
(d(a)b)(X) &= q^{m(g(a),g(b))} bda(X) = q^{m(g(a),g(b))+m(g(a),g(X))} bX(a) \\
&= q^{m(g(a),g(b)+g(X))+m(g(b),g(a)+g(X))} X(a)b \\
&= q^{m(g(a)+g(b),g(X))} X(a)b. \\
adb(X) &= q^{m(g(b),g(X))} aX(b).
\end{aligned}$$

Also we have $dx^i(\partial_j) = q^{m(g(x^i),g(\partial_j))}\partial_j(x^i) = q^{m(g(x^j),g(x^i))}\delta_i^j I = \delta_i^j I$. By Theorem 1, $X = X^i\partial_i$ for any $X \in \mathrm{Der}_q(\mathbb{C}_q^n)$. We have

$$\alpha(X^i\partial_i) = q^{m(g(\alpha),g(X^i))} X^i\alpha(\partial_i).$$

With $\alpha_i = \alpha(\partial_i)$ and $g(\alpha) = g(\alpha_i) - g(\partial_i)$ we obtain

$$\alpha(X^i\partial_i) = q^{m(g(\alpha),g(X^i))+m(g(X^i),g(x)+g(\partial_i))}\alpha_i X^i = q^{m(g(X^i),g(\partial_i))}\alpha_i X^i.$$

On the other hand,

$$\begin{aligned}
\alpha_j dx^j(X^i\partial_i) &= \alpha_j q^{m(g(x^j),g(X^i))} X^i dx^j(\partial_i) \\
&= q^{m(g(x^i),g(X^i))}\alpha_i X^i = q^{m(g(X^i),g(\partial_i))}\alpha_i X^i.
\end{aligned}$$

Therefore, $\alpha = \alpha(\partial_i)dx^i$ for homogeneous $\alpha$, thus also for nonhomogeneous $\alpha$.■

Thus, the $dx^i$ form a basis of the $\mathbb{C}_q^n$-bimodule $\bigwedge_{0q}^1(\mathbb{C}_q^n)$ as linear independence follows immediately from $dx^i(\partial_j) = \delta_j^i I$. The calculus presented here corresponds to $C_j^i(a) = \delta_j^i q^{m(g(a),g(x^i))}a$ for $a \in \mathbb{C}_q^{nh}$ in formula (1) of the introduction.

## Remarks

1. Following the same ideas as above one can introduce q-differential forms of higher than first degree on $\mathbb{C}_q^n$, and define analogues of the usual operations (exterior derivative, Lie derivative, inner derivative). In particular, the q-differential forms on $\mathbb{C}_q^n$ form a q-Graßmann algebra. This will be part of [6].

2. The calculus presented here can be extended to noncommutative tori. The algebra $\mathbb{T}_q^n$ corresponding to such a torus is obtained from $\mathbb{C}_q^n$ by first adding the inverses $x^{i^{-1}}$ and imposing further relations according to the rule (6). The resulting algebra consists of Laurent polynomials in $x^1, \ldots, x^n$. For $| q |= 1$ this algebra can be completed in a certain topology. This completion, being the algebra $\mathbb{T}_q^n$, is the algebra of Laurent series in $x^1, \ldots, x^n$ with fastly decreasing coefficients ([2, 8]). It is rather evident that the above notions are meaningful at least for the algebraic extension (see [6]).

3. From the viewpoint of quantum groups, $\mathbb{C}_q^n$ appears as a left comodule, and one would be interested in differential calculi adapted to this structure, i.e. left covariant differential calculi (see [12] for the definition of left and right covariance for calculi on a quantum group, the definition for comodules is analogous). Unfortunately, though our calculus looks very natural it is not left or right covariant with respect to the coactions of $\mathrm{M}_q(n)$ or $\mathrm{SU}_q(n)$ on $\mathbb{C}_q^n$. In [7] all $\mathrm{SU}_q(n)$-bicovariant calculi on $\mathbb{C}_q^n$ are classified (see also [11]). For $n \geq 3$ there are exactly two calculi, for $n = 2$ two one-parameter families of calculi. The two calculi given in [11] are the only $\mathrm{M}_q(2)$-bicovariant calculi on $\mathbb{C}_q^2$ ([10]).

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