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The hyperboloid as ordered symmetric space

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Abstract: We study the geometry of the one sheeted hyperboloid viewed as an ordered symmetric space. The geometric results are indispensible prerequesites for the harmonic analysis of causal integral equations on such spaces. The present note may be used as an introduction to the geometric parts of the two forthcoming papers [1, 2].

Consider $\mathbb{R}^3$ with the Lorentzian metric

$$(x_0, x_1, x_2) \cdot (y_0, y_1, y_2) = -x_0 y_0 + x_1 y_1 + x_2 y_2. \tag{1}$$

This metric induces an ordering on $\mathbb{R}^3$ via

$$x \geq y \Leftrightarrow x_0 \geq y_0 \quad \text{and} \quad (x_0 - y_0)^2 \geq (x_1 - y_1)^2 + (x_2 - y_2)^2. \tag{2}$$

In this note we study various properties of the hyperboloid

$$\mathbf{X} = \{x \in \mathbb{R}^3 \mid x \cdot x = 1\}$$

Figure 1

which are related to the ordering of $\mathbf{X}$ that comes from $\geq$. We identify $\mathbb{R}^3$ with $\mathfrak{sl}(2, \mathbb{R})$ via

$$(x_0, x_1, x_2) \leftrightarrow \begin{pmatrix} x_1 & x_0 + x_2 \\ -x_0 + x_2 & -x_1 \end{pmatrix}$$

so that the metric (1) gets identified with a multiple of the Cartan-Killing form of $\mathfrak{sl}(2,\mathbb{R})$. We let $G = \mathrm{Sl}(2,\mathbb{R})$ act on $\mathbf{X}$ via conjugation and fix a base point

$$e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the stabilizer H of e is given by

$$H = \{\begin{pmatrix} c & s \\ s & c \end{pmatrix} \mid c^2 - s^2 = 1\}$$

and $\mathbf{X} = G/H$.

The homogeneous space $\mathbf{X}$ carries the structure of a pseudo Riemannian symmetric space via the involution

$$\tau: G \to G$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

In fact, τ is just conjugation by

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and H is the group of fixed points G^τ. Note that

$$\theta: G \to G$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

is a Cartan involution commuting with τ. The group of its fixed elements

$$K = \{\begin{pmatrix} c & s \\ -s & c \end{pmatrix} \mid c^2 + s^2 = 1\}$$

is a maximal compact subgroup of G. On the level of Lie algebras we have

$$\theta\begin{pmatrix} x & y \\ z & -x \end{pmatrix} = \begin{pmatrix} -x & -z \\ -y & x \end{pmatrix}$$

$$\tau\begin{pmatrix} x & y \\ z & -x \end{pmatrix} = \begin{pmatrix} -x & z \\ y & x \end{pmatrix}$$

and

$$\mathfrak{k} = Lie(K) = \{\begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix} \mid y \in \mathbb{R}\}$$

$$\mathfrak{h} = Lie(H) = \{\begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix} \mid y \in \mathbb{R}\}.$$

The -1-eigenspaces of θ and τ are given by

$$\mathfrak{p} = \{\begin{pmatrix} x & y \\ y & -x \end{pmatrix} \mid x, y \in \mathbb{R}\}$$

$$\mathfrak{q} = \{\begin{pmatrix} x & y \\ -y & -x \end{pmatrix} \mid x, y \in \mathbb{R}\}.$$

In order to find a suitable Cartan decomposition of G we consider the maximal abelian subspace

$$\mathfrak{a} = \{\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \mid x \in \mathbb{R}\}$$

of $\mathfrak{p} \cap \mathfrak{q}$ (which is also maximal abelian in $\mathfrak{p}$ and $\mathfrak{q}$!) and set

$$A = \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t > 0\}$$

$$A^+ = \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t > 1\}.$$

Then

$$G = KA^+K \cup K \tag{3}$$

and

$$G = HTH \cup HAH \cup HwAH \cup \text{lower dimensional things.} \tag{4}$$

At this point we write T for K to stress the fact that K in our example is at the same time a maximal torus in G. The easiest way to see (4) is to consider the H-orbits on $\mathbf{X}$:

Figure 2

The correct analogon of (3) for $\mathbf{X}$ comes from the ordering: Note first that on $\mathbf{X}$ we have

$$x \geq y \Leftrightarrow x_0 \geq y_0 \quad \text{and} \quad x \cdot y \geq 1. \tag{2'}$$

Let

$$\mathbf{X}^+ = \{x \in \mathbf{X} \mid x \geq e\}.$$

Figure 3

If $\pi\colon G \to \mathbf{X}$ is the quotient map then $S = \pi^{-1}(\mathbf{X}^+)$ is a semigroup containing H. In fact, $g, g' \in S$ implies

$$\pi(gg') = g\pi(g') \geq ge = \pi(g) \geq e.$$

Let S^o be the interior of S then

$$S^o = HA^+H. \tag{5}$$

Next we consider a suitable Iwasawa decomposition KAN of G and its analogon HAN which will be just an open subset of G containing S. To this end we set

$$N = \{\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \mid r \in \mathbb{R}\}$$

$$\mathfrak{n} = \mathrm{Lie}(N) = \{\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \mid r \in \mathbb{R}\}$$

and note that

$$K \times A \times N \to KAN = G$$

$$(\begin{pmatrix} c & s \\ -s & c \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}) \mapsto \begin{pmatrix} ct & ctr + st{-}1 \\ -st & -str + ct^{-1} \end{pmatrix}$$

is a diffeomorphism. Similarly

$$H \times A \times N \to HAN \subseteq G$$

$$(\begin{pmatrix} c & s \\ s & c \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t{-}1 \end{pmatrix}, \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}) \mapsto \begin{pmatrix} ct & ctr + st{-}1 \\ st & str + ct{-}1 \end{pmatrix}$$

is a diffeomorphism onto an open subset of G. In order to show that $S \subseteq HAN$ we realize S as a semigroup of nonlinear contractions. Let

$$G^c = \mathrm{SU}(1,1) = \{\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \mid |a|^2 - |b|^2 = 1\}$$

$$K^c = (G^c)^\theta = \{\begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \mid |a| = 1\}.$$

Then

$$\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q} \quad \mathfrak{k}^c = \mathfrak{h} \cap \mathfrak{k} + i(\mathfrak{q} \cap \mathfrak{p})$$

and

$$G^c/K^c = D := \{z \in \mathbb{C} \mid |z| < 1\}$$

if we let G^c act on D via

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \cdot z = \frac{az+b}{\overline{b}z + \overline{a}}.$$

Both G and G^c have the same complexification $G_{\mathbb{C}} = \mathrm{Sl}(2,\mathbb{C})$. The hermitean symmetric space D can be viewed as G-orbit in the flag manifold

$$G_{\mathbb{C}}/K_{\mathbb{C}}^c \overline{P_{\mathbb{C}}^c} \cong \mathbb{P}_{\mathbb{C}}^1, \tag{6}$$

where

$$K_{\mathbb{C}}^c = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^\times\}$$

$$\overline{P_{\mathbb{C}}^c} = \{\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \mid c \in \mathbb{C}\}.$$

The isomorphy (6) is easy to see via fractional linear transformations

Figure 4

Note that τ induces a complex conjugation on D and $\mathbb{P}_{\mathbb{R}}^1$. Restricting the Borel embedding $D \to \mathbb{P}_{\mathbb{R}}^1$ to the real points yields

$$H/\{\pm 1\} \cong D_\tau = \{r \in \mathbb{R} \mid |r| < 1\} \to \mathbb{P}_{\mathbb{R}}^1 \cong G/A\overline{N},$$

where $\overline{N} = \tau N$. Using $D_\tau = H \cdot 0 = H \cdot D_\tau$ and the doubl e coset decomposition (4) one now finds

$$S = \{g \in G \mid g^{-1} \cdot D_\tau \subseteq D_\tau\} \tag{7}$$

which in turn implies $S \subseteq HA\overline{N}$ and then

$$S \subseteq HAN \tag{8}$$

because of $\tau(S) = S^{-1}$ (this follows from (5)) and $\tau(HA\overline{N}) = HAN$. In order to visualize the difference between S and HAN we consider

$$\mathbf{X}^{-} = S^{-1}/H \subseteq NAH/H \subseteq \mathbf{X}. \tag{8'}$$

The N-orbits on $\mathbf{X}$, described by

$$\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x + rz & -2xr - zr^2 + y \\ z & -zr - x \end{pmatrix}$$

simply are the parabolas which occur as intersections of $\mathbf{X}$ with the plane $\mathfrak{a} + \mathfrak{n}$. In analogy to the Riemannian case we call these parabolas horocycles:

Figure 5

For the harmonic analysis on the unit disk it is important to study Poisson integrals of functions on its boundary, the circle, which is a homogeneous space under the action of the maximal compact subgroup. These integrals can be written in terms of the Iwasawa projection $KAN \to A$. Note here that one should rather write an Iwasawa decomposition for G^c since it is not a general fact that

$$G/K \cong G^c/K^c \tag{9}$$

is also a bounded symmetric domain—such ordered symmetric spaces are called Cayley type spaces since the isomorphism (9) is effected by a Cayley transform. In our case K acts transitively on the boundary of G/K which gets identified with the homogeneous space $G/MAN = K/M = \mathbb{P}^1_{\mathbb{R}}$, where $M = \{\pm \mathbf{1}\}$ (this also can be se en via fractional linear transformations). This is contrasted by the four H-orbits on G/MAN:

Figure 6

The orbit $HMAN/MAN$ may be considered as a boundary of $\mathbf{X}^+$:

Figure 7

and the harmonic analysis on $\mathbf{X}^+$ depends heavily on the "Iwasawa projection" (this is why (8) is so important)

$$\begin{gathered} HAN \to A \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{a^2 - c^2} & 0 \\ 0 & \sqrt{a^2 - c^2}^{-1} \end{pmatrix} \end{gathered}$$

which should be compared to the usual Iwasawa projection

$$\begin{gathered} KAN \to A \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{a^2 + c^2} & 0 \\ 0 & \sqrt{a^2 + c^2}^{-1} \end{pmatrix}. \end{gathered}$$

References

[1] Faraut, J., J. Hilgert and G. 'Olafsson, *Harmonic analysis on ordered symmetric spaces*, to appear.

[2] Hilgert J. and K.H. Neeb, *Wiener-Hopf operators on ordered symmetric spaces*, to appear.

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