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Self-similar periodic tilings on the Heisenberg group

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Abstract. We construct a tiling on the Heisenberg group G with the following properties. A discrete cocompact subgroup of G acts freely and transitively on the set of tiles. Moreover, an expanding endomorphism of G carries each tile onto the union of k tiles, where k=4, and this is the least number for which such a construction is possible. Our computations are basic for the generation of arbitrary periodic self-similar tilings on G.

1. Introduction to tilings

Suppose that X is a complete metric space. A *tiling* is a locally finite family $\mathcal{T}$ of non-empty subsets of X with the following properties:

(i) for every $A \in \mathcal{T}$, $A = \mathrm{cl}(\mathrm{int}(A))$ (tiles are regular-closed),
(ii) $\mathrm{int}(A) \cap \mathrm{int}(B) = \emptyset$ for different tiles $A, B \in \mathcal{T}$ (non-overlap condition),
(iii) $\bigcup_{A\in\mathcal{T}} A = X$ (covering condition).

Let Γ be a discrete cocompact subgroup of the isometry group of X. We say that the tiling $\mathcal{T}$ is *periodic* under Γ (or a Γ-tiling) provided that for some tile A,

(iv) $\mathcal{T} = \{\gamma(A) : \gamma \in \Gamma\}$.

The distinguished tile A is then called the *prototile*.

Furthermore, assume that $\Phi : X \to X$ is an expanding map (i.e. stretching all distances by at least a factor $c > 1$), such that $\Phi\Gamma\Phi^{-1} \subset \Gamma$. A Γ-tiling $\mathcal{T}$ is *self-similar* iff

(v) for certain $\gamma_1, \ldots, \gamma_k \in \Gamma$, $\Phi(A) = \bigcup_{i=1}^{k} \gamma_i(A)$.

It follows that the elements γ_i form a complete set of right coset representatives of $\Phi\Gamma\Phi^{-1} \subset \Gamma$.

If the map Φ and the group elements γ_i are given then the tiling can be constructed as follows: one tile A is obtained as the attractor of the iterated function system $\{\Phi^{-1}\gamma_i : i = 1, \ldots, k\}$ (see Hutchinson [3]). It follows from a generalization of a theorem by Bandt [1] that the set A has non-empty interior, and hence one can construct a tiling by iterated expansion and subdivision of this set

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(cf. [3]). The tiling consists of copies of A under the action of a subset $\Gamma_0 \subset \Gamma$. We should mention that in general, Γ_0 may not be a group.

Recently, self-similar lattice tilings have been investigated. They are of special interest inspite of their relation to exotic number systems, Markov partitions, and wavelets; for examples, see [2], and the references there.

STRICHARTZ [5] has constructed wavelets on Lie groups of "Heisenberg type" but as a tiling framework, he used only one special type of tiling ("stacked tilings" over cubes). In our paper, we will work out the algebraic base for general self-similar periodic tilings of the Heisenberg group with respect to the subgroup of all elements with integer co-ordinates.

2. The Heisenberg Group and its Metric

The Heisenberg group is the group $G = \mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ where $\varphi : \mathbb{R} \to Aut(\mathbb{R}^2)$ is defined by

$$\varphi(z) = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}.$$

In other words, G is the topological space $\mathbb{R}^3$ with the multiplication formula

$$(x, y, z) \circ (x', y', z') = (x + x', y + y' + x'z, z + z').$$

A matrix representation of G is given by

$$(x, y, z) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix}.$$

The inverse element of (x, y, z) is $(x, y, z)^{-1} = (-x, xz - y, -z)$.
Let $L_{(x,y,z)}\colon G \to G$ be the left multiplication by (x, y, z). The derivative of this map is the linear map

$$DL_{(x,y,z)} = \begin{bmatrix} 1 & 0 & 0 \\ z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : \mathbb{R}^3 \to \mathbb{R}^3$$

at every point of G.
A metric $g : G \to (T^*G)^{\otimes 2}$ is left-invariant if for any $(x, y, z), (x', y', z') \in G$ and any two vectors $v, w \in T_{(x',y',z')}G$,

$$g_{(x',y',z')}(v, w) = g_{L_{(x,y,z)}(x',y',z')}(DL_{(x,y,z)}v, DL_{(x,y,z)}w)$$

is fulfilled. This means that the metric is determined by the choice of the scalar product at the identity of the group. Assume $g_0 = I$. Then we have

$$\begin{aligned} g_{(x,y,z)}(v, w) &= g_0(DL_{(x,y,z)^{-1}}v, DL_{(x,y,z)^{-1}}w) \\ &= \langle (v_1, v_2 - zv_1, v_3), (w_1, w_2 - zw_1, w_3) \rangle \\ &= (1 + z^2)v_1w_1 - z(v_1w_2 + v_2w_1) + v_2w_2 + v_3w_3, \end{aligned}$$

so the matrix form of the metric is

$$g_{(x,y,z)} = \begin{bmatrix} 1+z^2 & -z & 0 \\ -z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The volume of a subset $A \subset G$ is therefore

$$\mathrm{Vol}_g(A) = \int_A \sqrt{\det\, g_{(x,y,z)}}\, dx\, dy\, dz = \int_A dx\, dy\, dz = \mathrm{Vol}_{\mathrm{euclidean}}(A).$$

3. Endomorphisms of the Heisenberg group

Now we compute the general form of continuous endomorphisms $\Phi : G \to G$. Assume $\Phi(x,y,z) = (\xi(x,y,z), \eta(x,y,z), \zeta(x,y,z))$. Then the homomorphy condition,

$$\Phi(x,y,z) \circ \Phi(x',y',z') = \Phi((x,y,z) \circ (x',y',z')),$$

yields

$$\begin{aligned} \xi(x+x', y+y'+x'z, z+z') &= \xi(x,y,z) + \xi(x',y',z') \\ \eta(x+x', y+y'+x'z, z+z') &= \eta(x,y,z) + \eta(x',y',z') + \xi(x',y',z')\zeta(x,y,z) \\ \zeta(x+x', y+y'+x'z, z+z') &= \zeta(x,y,z) + \zeta(x',y',z'). \end{aligned}$$

Firstly we compute ξ. Denote $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. For the three one-parameter subgroups $\{te_i : t \in \mathbb{R}\}$ we obtain

$$\xi((s+t)e_i) = \xi(se_i) + \xi(te_i),$$

and it follows that

$$\xi(te_i) = \xi_i t$$

with real parameters ξ_i, $i = 1,2,3$. It follows that

$$\begin{aligned} \xi(x,y,z) &= \xi((x,0,0) \circ (0,y,0) \circ (0,0,z)) = \xi(x,0,0) + \xi(0,y,0) + \xi(0,0,z) \\ &= \xi_1 x + \xi_2 y + \xi_3 z. \end{aligned}$$

Moreover,

$$\begin{aligned} \xi(x', x'z, z) &= \xi((0,0,z) \circ (x',0,0)) = \xi(0,0,z) + \xi(x',0,0) \\ &= \xi(x',0,0) + \xi(0,0,z) = \xi((x',0,0) \circ (0,0,z)) = \xi(x',0,z)\,, \end{aligned}$$

hence $\xi_2 = 0$. Performing the same computation for ζ we obtain

$$\begin{aligned} \xi(x,y,z) &= \xi_1 x + \xi_3 z \\ \zeta(x,y,z) &= \zeta_1 x + \zeta_3 z \end{aligned}$$

for all $(x,y,z) \in G$. Now we compute η for the generating one-parameter groups. Obviously,

$$\eta(0,y,0) = \eta_2 y$$

for some real number η_2. For the groups $\{te_1\}$ and $\{te_3\}$ we have

$$\eta((s+t)e_i) = \eta(se_i) + \eta(te_i) + \xi_i\zeta_i st \qquad (i=1,3),$$

so we look for a continuous function f which satisfies the functional equation

$$f(s+t) = f(s) + f(t) + cst$$

with some constant c. By induction we obtain

$$f(mt) = m\,f(t) + \frac{m^2-m}{2}\,ct^2$$

for all natural numbers m and arbitrary real t. Assume $f(1)=a$. Then for $t=1$ the formula yields

$$f(x) = \frac{c}{2}\,x^2 + \left(a - \frac{c}{2}\right)x \qquad (*)$$

for $x \in I\!N$. For $t = 1/n$, n a positive integer, and $m = n$ we have

$$a = f\left(\frac{n}{n}\right) = n\,f\left(\frac{1}{n}\right) + \frac{n^2-n}{2}\,c\,\frac{1}{n^2}$$

which shows that $(*)$ holds for $x = 1/n$. Now let m be an arbitrary natural number and $t = 1/n$, then the formula is true for all rational numbers $x \geq 0$. By

$$0 = f(0) = f(x-x) = f(x) + f(-x) - c\,x^2$$

and the continuity of f we see that $(*)$ holds for all $x \in \mathbb{R}$. Hence,

$$\begin{aligned}\eta(x,0,0) &= \frac{\xi_1\zeta_1}{2}x^2 + \left(\eta_1 - \frac{\xi_1\zeta_1}{2}\right)x\\ \eta(0,0,z) &= \frac{\xi_3\zeta_3}{2}z^2 + \left(\eta_3 - \frac{\xi_3\zeta_3}{2}\right)z\end{aligned}$$

for certain constants η_1, η_3. Finally, the condition

$$\Phi(0,0,z) \circ \Phi(x,0,0) = \Phi(x, xz, z)$$

implies

$$\eta_2 = \xi_1\zeta_3 - \xi_3\zeta_1.$$

Now one checks that the formula

$$\Phi(x,y,z) = (\xi(x,y,z), \eta(x,y,z), \zeta(x,y,z))$$

with

$\xi(x,y,z) = \xi_1 x + \xi_3 z$

$\eta(x,y,z) = \frac{\xi_1\zeta_1}{2}x^2 + \left(\eta_1 - \frac{\xi_1\zeta_1}{2}\right)x + \frac{\xi_3\zeta_3}{2}z^2 + \left(\eta_3 - \frac{\xi_3\zeta_3}{2}\right)z + \xi_3\zeta_1 xz + (\xi_1\zeta_3 - \xi_3\zeta_1)y$

$\zeta(x,y,z) = \zeta_1 x + \zeta_3 z$

indeed defines an endomorphism of G. The derivative of this endomorphism is $D\Phi_(x,y,z) =$

$$\begin{bmatrix} \xi_1 & 0 & \xi_3 \\ \xi_1\zeta_1 x + \xi_3\zeta_1 z + \eta_1 - \frac{1}{2}\xi_1\zeta_1 & \xi_1\zeta_3 - \xi_3\zeta_1 & \xi_3\zeta_1 x + \xi_3\zeta_3 z + \eta_3 - \frac{1}{2}\xi_3\zeta_3 \\ \zeta_1 & 0 & \zeta_3 \end{bmatrix}.$$

Since we would like to construct self-similar tilings we must specify the conditions for Φ to be expanding. This is the case iff $D\Phi$ is expanding at every point. We have

$$\Phi = L_{\Phi(x,y,z)^{-1}}\,\Phi\, L_(x,y,z)$$

and hence

$$D\Phi_0 = DL_{\Phi(x,y,z)^{-1}}\, D\Phi_(x,y,z)\, DL_(x,y,z),$$

so the expansivity of Φ depends only on the choice of the scalar product at the identity. Since $DL_(x,y,z)$ and $DL_{\Phi(x,y,z)^{-1}}$ are isometries with respect to any left-invariant metric. But g_0 can be chosen in such a way that $D\Phi_0$ is expanding iff all eigenvalues of $D\Phi_0$ are outside the complex unit circle. Obviously, this is fulfilled iff the linear map with the matrix $\begin{bmatrix} \xi_1 & \xi_3 \\ \zeta_1 & \zeta_3 \end{bmatrix}$ is an expansion for some metric in $\mathbb{R}^2$.

4. The subgroup $\mathbb{Z}^2 \rtimes \mathbb{Z}$ and an example for a tiling

Consider the metric space G with the discrete group $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}$ consisting of all elements of G with integral coordinates, acting on G by left multiplication. This group is cocompact since one can easily verify that it holds $\Gamma \circ [0,1]^3 = G$.

It is obvious that an endomorphism of G preserves Γ iff all its parameters are integers. Moreover, the volume growth of Φ must be the same as the combinatorial growth factor; this means $k = |\det\ D\Phi|$. But $\det\ D\Phi = (\xi_1\zeta_3 - \xi_3\zeta_1)^2$, hence the minimal number of pieces a tile can have is 4.

Now we would like to construct a tile with 4 pieces. Let Φ be the endomorphism with parameters $\xi_1 = -\xi_3 = \zeta_1 = \zeta_3 = 1$, and $\eta_1 = \eta_3 = 0$, so

$$\Phi(x,y,z) = \left(x - z\,,\ \frac{x^2 - x - z^2 + z}{2} - xz + 2y\,,\ x + z\right).$$

For the right coset representatives take the group elements $\gamma_1 = (0,0,0)$, $\gamma_2 = (1,0,0)$, $\gamma_3 = (0,1,0)$, and $\gamma_4 = (1,1,0)$.

Notice that the set of coset representatives has a "bundle structure" in the following sense. For the group Γ there exists an exact sequence

$$0 \to \mathbb{Z} = \langle (0,1,0) \rangle \to \Gamma \to \mathbb{Z}^2 \to 0,$$

where $\mathbb{Z}^2$ is generated by the images of $(1,0,0)$ and $(0,0,1)$. The corresponding part of this sequence for the image of Φ is

$$0 \to 2\mathbb{Z} \to \Phi(\Gamma) \to \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbb{Z}^2 \to 0.$$

The sets $\{(0,0,0),(1,0,0)\}$ and $\{(0,0,0),(0,1,0)\}$ consist of coset representatives for the components contained in $\mathbb{Z}^2$ and $\mathbb{Z}$, respectively.
The set of all γ_i's is obtained as the product

$$\{(0,0,0),(1,0,0)\} \circ \{(0,0,0),(0,1,0)\}.$$

As a consequence, we obtain a "bundle" of tilings. We can construct two tilings, namely the wild twindragon tiling in $\mathbb{Z}^2$ (the "basis"), and the tiling by intervalls (related to the dual number system) on $\mathbb{R}$ (the "fibre"). The latter is obtained simply since Φ preserves the y-axis. For the twindragon tiling, the expansion is the superposition of Φ and the usual projection to the (x,z)-plane. The entire tile in G consists of continuum many line segments, glued together over the twindragon with varying heights.

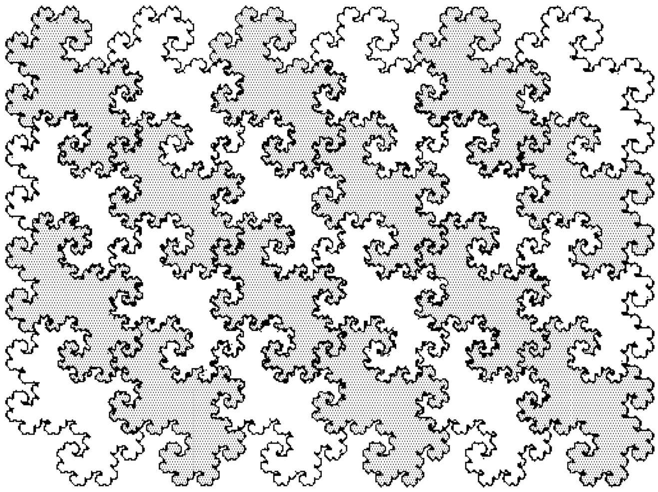

Fig. 1. Tiling by "wild twindragons". The expansion maps every tile onto a gray tile and its right white neighbor.

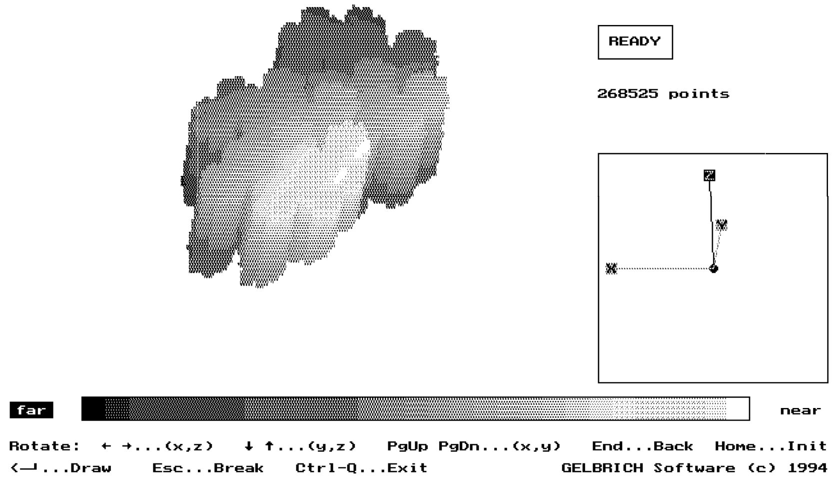

Fig. 2. A reptile on the Heisenberg group.

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