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PIUNIKHIN–SALAMON–SCHWARZ ISOMORPHISMS AND SPECTRAL INVARIANTS FOR CONORMAL BUNDLE

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Abstract. We give a construction of the Piunikhin–Salamon–Schwarz isomorphism between the Morse homology and the Floer homology generated by Hamiltonian orbits starting at the zero section and ending at the conormal bundle. We also prove that this isomorphism is natural in the sense that it commutes with the isomorphisms between the Morse homology for different choices of the Morse function and the Floer homology for different choices of the Hamiltonian. We define a product on the Floer homology and prove triangle inequality for conormal spectral invariants with respect to this product.

1. Introduction and main results

Let M be a compact smooth manifold. The cotangent bundle T^*M of M carries a natural symplectic structure $\omega = d\lambda$, where λ is the Liouville form. Let

$$\nu^*N = \{\alpha \in T_p^*M \mid p \in N,\, \alpha|_{T_pN} = 0\} \subset T^*M,$$

be a conormal bundle of a closed submanifold $N \subseteq M$. Let H be a time-dependent smooth compactly supported Hamiltonian on T^*M such that the intersection $\nu^*N \cap \phi_H^1(o_M)$ is transverse. Here, $\phi_H^t : T^*M \to T^*M$ denotes the Hamiltonian flow of the Hamiltonian vector field X_H. Floer chain groups $\mathrm{CF}_*(o_M, \nu^*N : H)$ are $\mathbb{Z}_2$-vector spaces generated by the finite set $\nu^*N \cap \phi_H^1(o_M)$ (see [**26**] for more details). The Floer homology $\mathrm{HF}_*(o_M, \nu^*N : H)$ is defined as the homology group of $(\mathrm{CF}_*(o_M, \nu^*N : H), \partial_F)$ where ∂_F is a boundary operator

$$\partial_F(x) = \sum_{y \in \nu^*N \cap \phi_H^1(o_M)} n(x, y; H)y,$$

and $n(x, y; H)$ is the (mod 2) number of solutions of the system

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$$
\begin{aligned}
&\frac{\partial u}{\partial s}+J\Big(\frac{\partial u}{\partial t}-X_H(u)\Big)=0, \quad u(-\infty,t)=\phi_H^t((\phi_H^1)^{-1})(x),\\
&u(s,0)\in o_M,\ u(s,1)\in\nu^*N, \quad u(+\infty,t)=\phi_H^t((\phi_H^1)^{-1})(y),\\
&\qquad\qquad x,y\in\nu^*N\cap\phi_H^1(o_M).
\end{aligned}\tag{1.1}
$$

This homology was introduced by Floer [**7**], developed by Oh [**23**] and Fukaya, Oh, Ohta and Ono in the most general case [**11**]. For a convenience, these groups will be denoted by $\mathrm{HF}_*(H)$. Although it is well known that these groups do not depend on H, we will keep H in the notation, since in many practical applications it is useful to keep track on the Hamiltonian used in their definition. For two regular pairs of parameters (H^α,J^α) and (H^β,J^β) the isomorphism $S^{\alpha\beta}:\mathrm{HF}_*(H^\alpha)\to\mathrm{HF}_*(H^\beta)$ between corresponding the Floer homology groups is induced by the chain homomorphism

$$
\sigma^{\alpha\beta}:\mathrm{CF}_*(H^\alpha)\to\mathrm{CF}_*(H^\beta),\ \ \sigma^{\alpha\beta}(x^\alpha)=\sum_{x^\beta}n(x^\alpha,x^\beta;H^{\alpha\beta})x^\beta,
$$

that counts the number $n(x^\alpha,x^\beta;H^{\alpha\beta})$ of solutions of the system

$$
\begin{aligned}
&\frac{\partial u}{\partial s}+J^{\alpha\beta}\Big(\frac{\partial u}{\partial t}-X_{H^{\alpha\beta}}(u)\Big)=0, \quad u(s,0)\in o_M,\ u(s,1)\in\nu^*N,\\
&u(-\infty,t)=\phi_{H^\alpha}^t((\phi_{H^\alpha}^1)^{-1})(x^\alpha), \quad x^\alpha\in\nu^*N\cap\phi_{H^\alpha}^1(o_M),\\
&u(+\infty,t)=\phi_{H^\beta}^t((\phi_{H^\beta}^1)^{-1})(x^\beta), \quad x^\beta\in\nu^*N\cap\phi_{H^\beta}^1(o_M).
\end{aligned}\tag{1.2}
$$

Here $H_s^{\alpha\beta}$ and $J_s^{\alpha\beta}$ are s-dependent families such that for some $R>0$

$$
H_s^{\alpha\beta}=\begin{cases}H^\alpha, & s\leqslant -R\\ H^\beta, & s\geqslant R,\end{cases}\qquad J_s^{\alpha\beta}=\begin{cases}J^\alpha, & s\leqslant -R\\ J^\beta, & s\geqslant R.\end{cases}
$$

We define the action functional $\mathcal{A}_H$ on the space of paths

$$
\Omega(o_M,\nu^*N)=\{\gamma:[0,1]\to T^*M\mid\gamma(0)\in o_M,\ \gamma(1)\in\nu^*N\}
$$

by $\mathcal{A}_H(\gamma)=-\int\gamma^*\lambda+\int_0^1H(\gamma(t),t)\,dt$. Critical points of $\mathcal{A}_H$ are Hamiltonian paths with ends on the zero section and the conormal bundle, i.e., $\mathrm{CF}_*(H)$. Now we can define filtered Floer homology. Denote $\mathrm{CF}_*^\lambda(H)=\mathbb{Z}_2\langle x\in\mathrm{CF}_*(H)\,|\,\mathcal{A}_H(x)<\lambda\rangle$. Since the action functional decreases along holomorphic strip (see [**23**] for details) the differential ∂_F preserves the filtration given by $\mathcal{A}_H$. Its restriction $\partial_F^\lambda=\partial_F|_{\mathrm{CF}_*^\lambda(H)}$ defines a boundary operator on the filtered complex $\mathrm{CF}_*^\lambda(H)$. The filtered Floer homology is now defined as the homology of the filtered complex

$$
\mathrm{HF}_*^\lambda(H)=H_*(\mathrm{CF}_*^\lambda(H),\partial_F^\lambda).
$$

Note that the filtered Floer homology depends on the Hamiltonian H.

Let us recall the definition of the Morse homology. For a Morse function $f:N\to\mathbb{R}$ the Morse chain complex, $\mathrm{CM}_*(N:f)$, is a $\mathbb{Z}_2$–vector space generated by the set of critical points of f. Morse homology groups $\mathrm{HM}_*(N:f)$ are the homology groups of $\mathrm{CM}_*(N:f)$ with respect to the boundary operator

$$
\partial_M:\mathrm{CM}_*(N:f)\to\mathrm{CM}_*(N:f),\quad \partial_M(p)=\sum_{q\in\mathrm{Crit}(f)}n(p,q;f)q,
$$

where $n(p, q; f)$ is the number of gradient trajectories that satisfy

$$\frac{d\gamma}{ds} = -\nabla f(\gamma), \quad \gamma(-\infty) = p, \ \gamma(+\infty) = q. \tag{1.3}$$

Here, γ is a negative g-gradient trajectory of f and g is a Riemannian metric on N such that (f, g) is the Morse–Smale pair. In a way analogous to $S^{\alpha\beta}$, we can define an isomorphism $T^{\alpha\beta} : \mathrm{HM}_*(f^\alpha) \to \mathrm{HM}_*(f^\beta)$ between Morse homologies of two different Morse functions f^α and f^β. For given Morse–Smale pairs (f^α, g^α) and (f^β, g^β), we choose a homotopy of the Riemannian metrics $g_s^{\alpha\beta}$ such that

$$g_s^{\alpha\beta} = \begin{cases} g^\alpha, & s \leqslant -R \\ g^\beta, & s \geqslant R. \end{cases}$$

The isomorphism $T^{\alpha\beta}$ is generated by the chain homomorphism

$$\tau^{\alpha\beta} : \mathrm{CM}_*(f^\alpha) \to \mathrm{CM}_*(f^\beta) \ \text{ where } \ \tau^{\alpha\beta}(p^\alpha) = \sum_{p^\beta} n(p^\alpha, p^\beta; f^{\alpha\beta}) p^\beta,$$

that counts the number $n(p^\alpha, p^\beta; f^{\alpha\beta})$ of solutions of the system

$$\frac{d\gamma}{ds} = -\nabla_{g_s^{\alpha\beta}} f^{\alpha\beta}(\gamma), \quad \gamma(-\infty) = p^\alpha, \ \gamma(+\infty) = p^\beta, \tag{1.4}$$

(see [**32**] for details). We use a brief notation $\mathrm{HM}_*(f)$ or $\mathrm{HM}_*(N)$ instead of $\mathrm{HM}_*(N : f)$. Morse homology groups $\mathrm{HM}_*(f)$ are isomorphic to singular homology groups $H_*(N; \mathbb{Z}_2)$ [**21, 29, 32**] (we will sometimes identify Morse and singular homologies).

Our first theorem gives isomorphisms between the Morse homology $\mathrm{HM}_*(N : f)$ and the Floer homology $\mathrm{HF}_*(o_M, \nu^*N : H)$. These isomorphisms are essentially different from ones defined in [**26**].

Theorem 1.1. *There exist isomorphisms*

$$\Phi : \mathrm{HF}_k(o_M, \nu^*N : H) \to \mathrm{HM}_k(N : f),$$
$$\Psi : \mathrm{HM}_k(N : f) \to \mathrm{HF}_k(o_M, \nu^*N : H),$$

that are inverse to each other: $\Phi \circ \Psi = \mathbb{I}|_{HM}$ *and* $\Psi \circ \Phi = \mathbb{I}|_{HF}$.

In order to obtain isomorphisms on homology level, we consider homomorphisms on chain complexes defined by counting the intersection number of the space of gradient trajectories of function f and the space of perturbed holomorphic discs with boundary on the zero section o_M and the conormal bundle ν^*N (see Figure 1).

The main problem we need to overcome is that we have singular Lagrangian boundary conditions on holomorphic discs since an intersection $o_M|_N = o_M \cap \nu^*N$ is not transverse.

Motivation for this isomorphism was the paper by Piunikhin, Salamon and Schwarz [**25**], where they considered the Floer homology for periodic orbits, and the paper by Katić and Milinković [**15**], where they gave a construction of Piunikhin–Salamon–Schwarz isomorphisms in Lagrangian intersections Floer homology for a cotangent bundle. They worked with the Floer homology generated by Hamiltonian

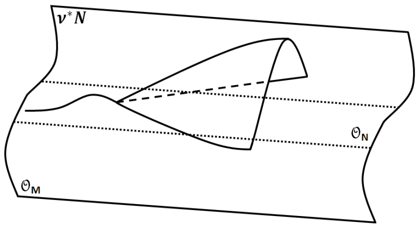

FIGURE 1. Intersection of gradient trajectory and perturbed holomorphic disc

orbits that start and end on zero section o_M. We obtain that isomorphism as special case for $N = M$. Albers [**2**] constructed a PSS-type homomorphism (which is not necessarily an isomorphism) in a more general symplectic manifold.

In [**26**] Poźniak constructed a different type of isomorphism between the Morse homology $\mathrm{HM}_*(N : f)$ and the Floer homology $\mathrm{HF}_*(o_M, \nu^*N : H_f)$. Namely, he used Hamiltonian H_f that is an extension of a Morse function f. We do not have that kind of restriction, our Hamiltonian H does not have to be an extension of a Morse function f.

Another advantage of using our isomorphism is its naturalness. When using Poźniak's type isomorphism, it is not obvious whether the diagram

$$\begin{tikzcd} \mathrm{HF}_*(H^\alpha) \arrow[r, "S^{\alpha\beta}"] & \mathrm{HF}_*(H^\beta) \\ \mathrm{HM}_*(f^\alpha) \arrow[u] \arrow[r, "T^{\alpha\beta}"] & \mathrm{HM}_*(f^\beta) \arrow[u] \end{tikzcd}$$

commutes, because different type of equations are used in definitions of $S^{\alpha\beta}$ and $T^{\alpha\beta}$. If we use our, PSS–type, isomorphisms as vertical arrows, then we obtain commutativity of the diagram above.

THEOREM 1.2. *The diagram*

$$\begin{tikzcd} \mathrm{HF}_k(o_M, \nu^*N : H^\alpha) \arrow[r, "S^{\alpha\beta}"] & \mathrm{HF}_k(o_M, \nu^*N : H^\beta) \\ \mathrm{HM}_k(N : f^\alpha) \arrow[u, "\Psi^\alpha"'] \arrow[r, "T^{\alpha\beta}"] & \mathrm{HM}_k(N : f^\beta), \arrow[u, "\Psi^\beta"'] \end{tikzcd}$$

commutes.

Using the existence of PSS isomorphism, we can define conormal spectral invariants and prove some of their properties. Denote by $\imath^\lambda_* : \mathrm{HF}^\lambda_*(H) \to \mathrm{HF}_*(H)$ the homomorphism induced by the inclusion map $\imath^\lambda : \mathrm{CF}^\lambda_*(H) \to \mathrm{CF}_*(H)$. For $\alpha \in \mathrm{HM}_*(N : f)$ define a conormal spectral invariant

$$l(\alpha; o_M, \nu^*N : H) = \inf\{\lambda \mid \Psi(\alpha) \in \mathrm{im}(\imath^\lambda_*)\}.$$

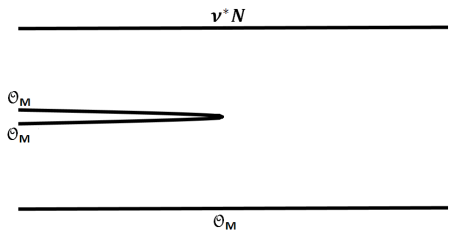

FIGURE 2. Pair–of–pairs object that defines the product $\star$

Oh defined Lagrangian spectral invariants in [**23**] using the idea of Viterbo's invariants for generating functions (see [**34**]). It turns out that those two invariants are the same (under some normalizaton conditions), see [**19, 20**].

Following [**3**], we can define a natural homology action homomorphism of $\mathrm{HF}_*(o_M, o_M)$ on $\mathrm{HF}_*(o_M, \nu^*N)$. Note that $\mathrm{HF}_*(o_M, o_M)$ stands for the Floer homology for conormal bundle in a special case when $M = N$. This is a standard product in Lagrangian Floer homology. Moreover, we can relate it, via the PSS isomorphism, to the action on the Morse side where it becomes the action of $\mathrm{HM}_*(M)$ on $\mathrm{HM}_*(N)$ via the external intersection product. As a result we obtain a triangle inequality for spectral invariants.

THEOREM 1.3. *Let $H_1, H_2, H_3 \in C_c^\infty([0,1] \times T^*M)$ be three Hamiltonians with a compact support. Then, there exists a natural homology action homomorphism*

$$\star : \mathrm{HF}_*(o_M, o_M : H_1) \otimes \mathrm{HF}_*(o_M, \nu^*N : H_2) \to \mathrm{HF}_*(o_M, \nu^*N : H_3).$$

The product $\star$, via the PSS, induces the exterior intersection product on the Morse homology

$$\cdot : \mathrm{HM}_*(M) \otimes \mathrm{HM}_*(N) \to \mathrm{HM}_*(N),$$

i.e., for $\alpha \in \mathrm{HM}_*(M)$ and $\beta \in \mathrm{HM}_*(N)$ it holds $\Psi(\alpha \cdot \beta) = \Psi(\alpha) \star \Psi(\beta)$.

Spectral invariants are subadditive with respect to the exterior intersection product, for $\alpha \in \mathrm{HM}_*(M)$ and $\beta \in \mathrm{HM}_*(N)$ such that $\alpha \cdot \beta \neq 0$ it holds

$$l(\alpha \cdot \beta; o_M, \nu^*N : H_1 \sharp H_2) \leqslant l(\alpha; o_M, o_M : H_1) + l(\beta; o_M, \nu^*N : H_2). \tag{1.5}$$

For the sake of completeness, we provide a construction of $\star$ in Section 5 although it is well known. This product is defined by counting a pair-of-pants with appropriate boundary conditions (see Figure 2). The exterior intersection product in Morse homology is defined by counting gradient trees of appropriate Morse functions (see Section 5 for the definition). The notion of the exterior intersection product was studied in [**5**], Subsection 4.3.

If we put $\alpha = [M]$ ($[M]$ is the fundamental class) and $H_2 = 0$ in (1.5), then we conclude that conormal spectral invariants are bounded for every nonzero singular homology class. The idea of this property came from Humilière, Leclercq and Seyfaddini's paper [**13**]. Note that the concatenation $H \sharp 0$ is just a reparametrization of H and it does not change Hamiltonian orbits, Floer strip or spectral invariants.

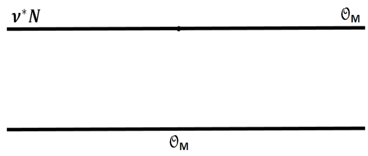

FIGURE 3. Holomorphic strip with a jump that defines the inclusion morphism

COROLLARY 1.1. *For every* $\alpha \in \mathrm{HM}_*(N) \smallsetminus \{0\}$ *it holds*

$$l(\alpha; o_M, \nu^* M : H) \leqslant l([M]; o_M, o_M : H).$$

Observing perturbed holomorphic strips with a jump on the upper boundary (see Figure 3), we can define the inclusion morphism of the Floer homologies. Using the PSS isomorphism, we obtain the inclusion morphism on the Morse side and the appropriate inequality among spectral invariants.

THEOREM 1.4. *Let* $H \in C_c^\infty([0,1] \times T^*M)$ *be a compactly supported Hamiltonian. There exists a morphism* $m : \mathrm{HF}_*(o_M, \nu^*N : H) \to \mathrm{HF}_*(o_M, o_M : H)$ *in Floer homology. On Morse homology level it holds* $\Phi \circ m \circ \Psi = i_*$, *where* i_* *is the morphism induced by the inclusion* $i : N \hookrightarrow M$ *in the sense of Schwarz* [**32**, Auxiliary Proposition 4.22]. *This gives rise to the following inequality among spectral invariants*

$$l(i_*(\alpha); o_M, o_M : H) \leqslant l(\alpha; o_M, \nu^* N : H), \tag{1.6}$$

for every $\alpha \in \mathrm{HM}_*(N) \smallsetminus \{0\}$.

Inequality (1.6) is expected because of the next observation. If α is realized at level λ in the filtered Lagrangian Floer homology $\mathrm{HF}_*^\lambda(o_M, \nu^*N)$, then it is also realized, via the inclusion, at the same level, in the homology $\mathrm{HF}_*^\lambda(o_M, o_M)$.

It is obvious that the composition of morphisms $\star$ and m lead to the product on Lagrangian Floer homology. Via the PSS, we obtain the operation on Morse homology.

COROLLARY 1.2. *Let* $H_1, H_2, H_3 \in C_c^\infty([0,1] \times T^*M)$ *be three Hamiltonians with compact support. Then, there exists a product*

$$* : \mathrm{HF}_*(o_M, \nu^*N : H_1) \otimes \mathrm{HF}_*(o_M, \nu^*N : H_2) \to \mathrm{HF}_*(o_M, \nu^*N : H_3),$$

in homology, defined by $* = \star \circ (m \otimes \mathbb{I})$. *The product* $*$ *induces the operation on* $\mathrm{HM}_*(N)$ *via the PSS isomorphism as* $\alpha \bullet \beta = \Phi(\Psi(\alpha) * \Psi(\beta))$, *for* $\alpha, \beta \in \mathrm{HM}_*(N)$.

As a special case, when $N = M$, we obtain the product defined in [**24**] (also discussed in [**16**]). We can see that $*$ counts pair-of-pants with a boundary on $o_M \cup \nu^*N$ and a jump from o_M to ν^*N on a slit of pants (see Figure 4). The operation $\bullet$ on $\mathrm{HM}_*(N)$ can be described as a composition of the inclusion and the exterior intersection product.

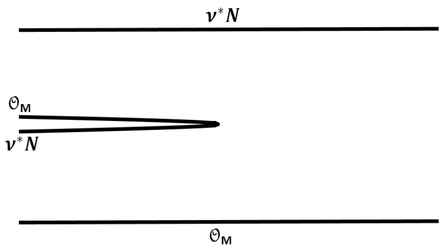

FIGURE 4. Pair–of–pants object that defines product on $\mathrm{HF}_*(o_M, \nu^*N : H)$

The triangle inequality for conormal spectral invariant, with respect to $\bullet$, follows directly from Theorem 1.3 and Theorem 1.4. Our inequality is a generalization of the one made by Monzner, Vichery and Zapolsky in [**22**].

COROLLARY 1.3. *Let us take two compactly supported Hamiltonians* H, H' *and* $\alpha, \beta \in \mathrm{HM}_*(N)$ *such that* $\alpha \bullet \beta \neq 0$. *Then*

$$l(\alpha \bullet \beta; o_M, \nu^*N : H\sharp H') \leqslant l(\alpha; o_M, \nu^*N : H) + l(\beta; o_M, \nu^*N : H').$$

This paper is organized as follows. In Section 2, we define diverse moduli spaces and prove some of their properties. In Section 3, we present the construction of PSS-type homomorphisms and we prove Theorem 1.1. Section 4 contains a proof of Theorem 1.2. In the last section, we provide constructions of morphisms $\star$ and m, and prove the mentioned inequalities among spectral invariants.

2. Holomorphic discs, gradient trajectories and moduli spaces

We start with a construction of mixed-type object space that we use for the definition of Ψ and Φ. Let p be a critical point of a Morse function f. Morse homology $\mathrm{HM}_k(f)$ is graded by Morse index $k = m_f(p)$ of critical points.

To each element of $\mathrm{CF}_*(H)$, we can assign a solution of the Hamiltonian equation

$$\dot{x} = X_H(x), \quad x(0) \in o_M, \quad x(1) \in \nu^*N. \tag{2.1}$$

For a solution x of (2.1), there exists a canonically assigned Maslov index

$$\mu_N : \mathrm{CF}_*(H) \to \tfrac{1}{2}\mathbb{Z},$$

see [**23, 27, 28**] for details. The Floer homology $\mathrm{HF}_k(H)$ is graded by $k = \mu_N(x) + \frac{1}{2}\dim N$.

Let $\mathcal{M}(p, f, g; x, H, J)$ be the space of pairs of maps

$$\gamma : (-\infty, 0] \to N, \quad u : \mathbb{R} \times [0, 1] \to T^*M,$$

that satisfy

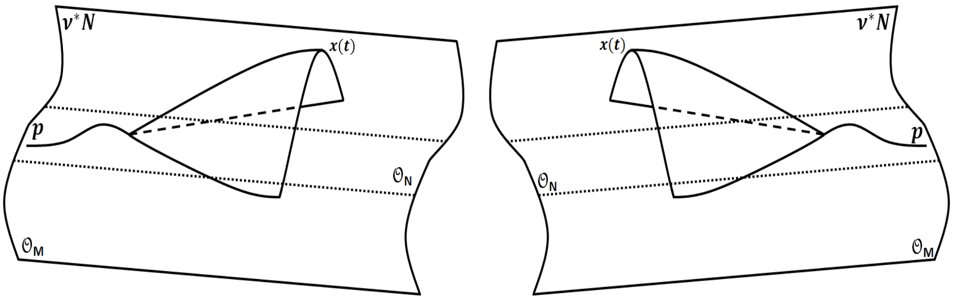

FIGURE 5. $\mathcal{M}(p, f, g; x, H, J)$ *and* $\mathcal{M}(x, H, J; p, f, g)$

$$\begin{aligned}
&\frac{d\gamma}{ds} = -\nabla f(\gamma(s)), \quad \frac{\partial u}{\partial s} + J\Big(\frac{\partial u}{\partial t} - X_{\rho_R^+ H}(u)\Big) = 0,\\
&E(u) = \iint_{\mathbb{R}\times[0,1]} \|\partial_s u\|_J^2 \, dt\, ds < +\infty,\\
&u(s,0) \in o_M,\ u(s,1) \in \nu^*N,\ s \in \mathbb{R},\\
&\gamma(-\infty) = p, \ \ u(+\infty, t) = x(t), \ \ \gamma(0) = u(-\infty),
\end{aligned}$$

where R is a positive fixed number and $\rho_R^+ : \mathbb{R} \to \mathbb{R}$ is a smooth function such that

$$\rho_R^+(s) = \begin{cases} 1, & s \geqslant R+1 \\ 0, & s \leqslant R. \end{cases}$$

The strip u is holomorphic for $s \leqslant R$ and has finite energy. So, u admits a unique continuous extension $u(-\infty)$ (see [**18**, Section 4.5] and [**31**, Theorem 3.1]). The extension is a point that belongs to $o_N = \nu^*N \cap o_M$, and we can omit the second argument of $u(-\infty)$.

Let $\mathcal{M}(x, H, J; p, f, g)$ be the space of pairs of maps

$$\gamma : [0, +\infty) \to N, \quad u : \mathbb{R} \times [0,1] \to T^*M,$$

that satisfy

$$\begin{aligned}
&\frac{d\gamma}{ds} = -\nabla f(\gamma(s)), \quad \frac{\partial u}{\partial s} + J\Big(\frac{\partial u}{\partial t} - X_{\rho_R^- H}(u)\Big) = 0,\\
&E(u) < +\infty, \ \ u(s,0) \in o_M, \ \ u(s,1) \in \nu^*N, \ \ s \in \mathbb{R},\\
&\gamma(+\infty) = p, \ \ u(-\infty, t) = x(t), \ \ \gamma(0) = u(+\infty),
\end{aligned}$$

where $\rho_R^- : \mathbb{R} \to \mathbb{R}$ is a smooth function such that

$$\rho_R^-(s) = \begin{cases} 1, & s \leqslant -R-1 \\ 0, & s \geqslant -R. \end{cases}$$

PROPOSITION 2.1. *For a generic Morse function f and a generic compactly supported Hamiltonian H, the set $\mathcal{M}(p, f, g; x, H, J)$ is a smooth manifold of dimension $m_f(p) - (\mu_N(x) + \frac{1}{2}\dim N)$, and $\mathcal{M}(x, H, J; p, f, g)$ is a smooth manifold of dimension $\mu_N(x) + \frac{1}{2}\dim N - m_f(p)$.*

PROOF. Let $W^u(p, f)$ be the unstable manifold associated to a critical point p of a Morse function f. We know that $\dim W^u(p, f) = m_f(p)$ [**21**].

Let $\mathcal{M}_+(H, J; x)$ be the set of solutions of

$$u : \mathbb{R} \times [0,1] \to T^*M, \quad \frac{\partial u}{\partial s} + J\Big(\frac{\partial u}{\partial t} - X_{\rho_R^+ H}(u)\Big) = 0, \quad E(u) < +\infty,$$
$$u(s, 0) \in o_M, \; u(s,1) \in \nu^*N, \; s \in \mathbb{R}, \quad u(+\infty, t) = x(t).$$

The dimension of $\mathcal{M}_+(H, J; x)$ is $\dim \mathcal{M}_+(H, J; x) = \frac{1}{2}\dim N - \mu_N(x)$, see [**23**] for details. We used the definition of Maslov index $\mu_N(x) = \mu(B_\Phi(\mathbb{R}^m), V^\Phi)$, where $\Phi : x^*T(T^*M) \to [0,1] \times \mathbb{C}^m$ is any trivialization and

$$V^\Phi = \Phi(T_{x(1)}\nu^*N), \quad B_\Phi(t) = \Phi \circ T\phi_H^t \circ \Phi^{-1}.$$

For a generic choice of parameters, the evaluation map

$$Ev : W^u(p, f) \times \mathcal{M}_+(H, J; x) \to N \times N, \quad Ev(\gamma, u) = (\gamma(0), u(-\infty)),$$

is transversal to the diagonal, thus $\mathcal{M}(p, f, g; x, H, J) = Ev^{-1}(\triangle)$ is a smooth manifold of dimension

$$m_f(p) + \tfrac{1}{2}\dim N - \mu_N(x) - (2\dim N - \dim N) = m_f(p) - \tfrac{1}{2}\dim N - \mu_N(x).$$

The proof for $\mathcal{M}(x, H, J; p, f, g)$ is similar. □

We need some additional properties of the manifolds $\mathcal{M}(p, f, g; x, H, J)$ and $\mathcal{M}(x, H, J; p, f, g)$. The set of solutions of (1.1) is denoted by $\mathcal{M}(x, y; H)$ and $\mathcal{M}(p, q; f)$ denotes the set of solutions of (1.3) (modulo $\mathbb{R}$-action).

PROPOSITION 2.2. *Let f be a generic Morse function and H a generic compactly supported Hamiltonian. If $m_f(p) = \mu_N(x) + \frac{1}{2}\dim N$, then $\mathcal{M}(p, f, g; x, H, J)$ is a finite set. If $m_f(p) = \mu_N(x) + \frac{1}{2}\dim N + 1$, then $\mathcal{M}(p, f, g; x, H, J)$ is one-dimensional manifold with topological boundary*

$$\partial\mathcal{M}(p, f, g; x, H, J) = \bigcup_{m_f(q) = m_f(p) - 1} \mathcal{M}(p, q; f) \times \mathcal{M}(q, f, g; x, H, J)$$
$$\cup \bigcup_{\mu_N(y) = \mu_N(x) + 1} \mathcal{M}(p, f, g; y, H, J) \times \mathcal{M}(y, x; H).$$

PROOF. Let (γ_n, u_n) be a sequence in $\mathcal{M}(p, f, g; x, H, J)$ that has no $W^{1,2}$-convergent subsequence. Since N is compact, $\gamma_n(t)$ is bounded for every t. The sequence γ_n is equicontinuous because

$$d(\gamma_n(t_1), \gamma_n(t_2)) \leqslant \int_{t_1}^{t_2} \|\dot\gamma(s)\|\, ds$$

$$\begin{aligned}&\leqslant \sqrt{t_2-t_1}\sqrt{\int_{t_1}^{t_2}\|\dot\gamma(s)\|^2\,ds} = \sqrt{t_2-t_1}\sqrt{\int_{t_1}^{t_2}\frac{\partial}{\partial s}f(\gamma_n(s))\,ds}\\ &\leqslant \sqrt{t_2-t_1}\sqrt{\max_{x\in N} f(x)-f(\gamma_n(-\infty))} = \sqrt{t_2-t_1}\sqrt{\max_{x\in N} f(x)-f(p)}.\end{aligned}$$

It follows from the Arzelà–Ascoli theorem that γ_n has a subsequence converging uniformly on compact sets. Since the sequence γ_n is a solution of the equation $\dot\gamma_n = -\nabla f(\gamma_n)$, and the function f is smooth, γ_n converges with all its derivatives on compact subsets of $(-\infty, 0]$.

The energy of u_n is uniformly bounded since

$$\begin{aligned}\mathcal{A}_H(x(t)) &= \mathcal{A}_{\rho_R^+H}(u_n(+\infty), t) - \mathcal{A}_{\rho_R^+H}(u_n(-\infty), t) =\\ &= -E(u_n) + \int_{-\infty}^{+\infty}\int_0^1 (\rho_R^+(s))' H(u_n(s,t),t)\,dt\,ds.\end{aligned}$$

The Hamiltonian H has a compact support, $(\rho_R^+(s))'$ is nonzero only on $[R, R+1]$, so the last integral is uniformly bounded

$$\left|\int_{-\infty}^{+\infty}\int_0^1 (\rho_R^+(s))' H(u_n(s,t),t)\,dt\,ds\right| \leqslant C.$$

We have a sequence u_n whose energy is uniformly bounded. From the Gromov compactness [**12**], it follows that u_n has a subsequence that converges together with all derivatives on compact subsets of $(\mathbb{R}\times[0,1]) \smallsetminus \{z_1, \dots, z_m\}$. Bubbles can occur at z_i if it is an interior point of $\mathbb{R}\times[0,1]$. It is also possible that a bubble appears at the boundary point z_k as holomorphic disc with the boundary conditions on zero section and conormal bundle. But in our case neither holomorphic spheres nor discs appear. If $v : S^2 \to T^*M$ is a holomorphic sphere, then

$$\int_{S^2}\|dv\|^2 = \int_{S^2} v^*\omega = \int_{\partial S^2} v^*\lambda = 0.$$

If $v : \mathbb{R}\times[0,1] \to T^*M$ is a holomorphic disc, then

$$\int_{\mathbb{R}\times[0,1]}\|dv\|^2 = \int_{\mathbb{R}\times[0,1]} v^*\omega = \int_{\partial(\mathbb{R}\times[0,1])} v^*\lambda = 0,$$

since $\lambda = 0$ on o_M and ν^*N.

So, (γ_n, u_n) has a subsequence which converges with all its derivatives uniformly on compact sets. From C^∞_{loc}-convergence it follows $W^{1,2}$-convergence. Thus, (γ_n, u_n) has a subsequence that converges to some element of $\mathcal{M}(p^m, f, g; x^0, H, J)$. Similarly as in [**8, 14, 17, 30, 32**], we conclude that the only loss of compactness is a “trajectory breaking" in the following way

$$\begin{aligned}\bigcup \mathcal{M}(p, p^1; f) \times \cdots \times \mathcal{M}(p^{m-1}, p^m; f) \times \mathcal{M}(p^m, f, g; x^0, H, J)&\\ \times \mathcal{M}(x^0, x^1; H) \times \cdots \times \mathcal{M}(x^{l-1}, x; H).&\end{aligned} \tag{2.2}$$

Here, $p, p^1, \dots, p^m$ are critical points of f and $x^0, \dots, x^{l-1}, x$ are Hamiltonian paths with decreasing Morse and Maslov indices such that $m_f(p^m) \geqslant \mu_N(x^0) + \frac{1}{2}\dim N$.

Therefore, we have that the boundary $\partial\mathcal{M}(p,f,g;x,H,J)$ is a subset of union (2.2). The other inclusion follows from the standard gluing arguments.

If $m_f(p)=\mu_N(x)+\frac{1}{2}\dim N$, then $\mathcal{M}(p,f,g;x,H,J)$ is a compact, zero-dimensional manifold, so $\mathcal{M}(p,f,g;x,H,J)$ has a finite number of elements.

If $m_f(p)=\mu_N(x)+\frac{1}{2}\dim N+1$ then the boundary of $\mathcal{M}(p,f,g;x,H,J)$ can contain an element of a set $\mathcal{M}(p,q;f)\times\mathcal{M}(q,f,g;x,H,J)$ for some $q\in\mathrm{Crit}(f)$ such that $m_f(q)=m_f(p)-1$ or an element of a set $\mathcal{M}(p,f,g;y,H,J)\times\mathcal{M}(y,x;H)$ for some Hamiltonian orbit y, such that $\mu_N(y)=\mu_N(x)+1$. □

We have a similar proposition for $\mathcal{M}(x,H,J;p,f,g)$.

Proposition 2.3. *Let f be a generic Morse function and H a generic compactly supported Hamiltonian. If $m_f(p)=\mu_N(x)+\frac{1}{2}\dim N$, then $\mathcal{M}(x,H,J;p,f,g)$ is a finite set. If $m_f(p)=\mu_N(x)+\frac{1}{2}\dim N-1$, then $\mathcal{M}(x,H,J;p,f,g)$ is one-dimensional manifold with topological boundary*

$$\partial\mathcal{M}(x,H,J;p,f,g)=\bigcup_{m_f(q)=m_f(p)+1}\mathcal{M}(x,H,J;q,f,g)\times\mathcal{M}(q,p;f)$$
$$\cup\bigcup_{\mu_N(y)=\mu_N(x)-1}\mathcal{M}(x,y;H)\times\mathcal{M}(y,H,J;p,f,g).$$

Now, we define some auxiliary manifolds that we use to prove that the composition $\Phi\circ\Psi$ is the identity (see Theorem 1.1). Let $R>0$ be a fixed number. For $p,q\in\mathrm{Crit}(f)$ we define $\mathcal{M}_R(p,q,f;H)$ as the set of maps

$$\gamma_-:(-\infty,0]\to N,\ \ \gamma_+:[0,+\infty)\to N,\ \ u:\mathbb{R}\times[0,1]\to T^*M$$

such that

$$\frac{d\gamma_\pm}{ds}=-\nabla f(\gamma_\pm),\ \ \frac{\partial u}{\partial s}+J\Big(\frac{\partial u}{\partial t}-X_{\sigma_R H}(u)\Big)=0,\ \ E(u)<+\infty,$$
$$\gamma_-(-\infty)=p,\ \gamma_+(+\infty)=q,\ \ u(s,0)\in o_M,\ u(s,1)\in\nu^*N,\ s\in\mathbb{R},$$
$$u(\pm\infty,t)=\gamma_\pm(0),$$

where $\sigma_R:\mathbb{R}\to[0,1]$ is a smooth function such that

$$\sigma_R(s)=\begin{cases}1, & |s|\leqslant R\\ 0, & |s|\geqslant R+1.\end{cases}$$

We also define its parameterized version

$$\overline{\mathcal{M}}(p,q,f;H)=\big\{(R,\gamma_-,\gamma_+,u)\,|\,(\gamma_-,\gamma_+,u)\in\mathcal{M}_R(p,q,f;H),\ R>R_0\big\},$$

(see Figure 6). From now on, whenever we define new moduli space, we omit the argument g and J, although we know that a moduli space depend on a Riemannian metric and on an almost complex structure. For a generic choice of parameters, the set $\overline{\mathcal{M}}(p,q,f;H)$ is an one-dimensional manifold if $m_f(p)=m_f(q)$, and a zero-dimensional manifold if $m_f(p)=m_f(q)-1$.

Knowing the definitions of a broken gradient trajectory and a weak convergence of gradient trajectories [**32**], we can define a broken holomorphic strip and a weak convergence of holomorphic strips [**30**].

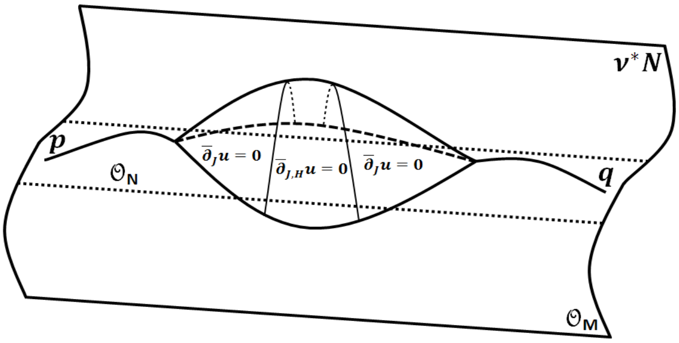

FIGURE 6. $\mathcal{M}_R(p, q, f; H)$

DEFINITION 2.1. A broken (perturbed) holomorphic strip v is a pair (v_1, v_2) of (perturbed) holomorphic strips such that $v_1(+\infty, t) = v_2(-\infty, t)$. A sequence of perturbed holomorphic strips $u_n : \mathbb{R} \times [0, 1] \to T^*M$ is said to converge weakly to a broken trajectory v if there exists a sequence of translations $\varphi_n^i : \mathbb{R} \times [0, 1] \to \mathbb{R} \times [0, 1]$, $i = 1, 2$, such that $u_n \circ \varphi_n^i$ converges to v_i uniformly with all derivatives on a compact subset of $\mathbb{R} \times [0, 1]$. We say that an element of mixed type (γ, u) is a broken element if γ is a broken trajectory or u is a broken holomorphic strip.

The following proposition gives us a boundary of a one-dimensional manifold $\overline{\mathcal{M}}(p, q, f; H)$.

PROPOSITION 2.4. *Let* $p, q \in \mathrm{CM}_k(f)$. *The topological boundary of* $\overline{\mathcal{M}}(p, q, f; H)$ *can be identified with*

$$\begin{aligned}\partial\overline{\mathcal{M}}(p, q, f; H) = \mathcal{M}_{R_0}(p, q, f; H) &\cup \bigcup_{m_f(r)=k-1} \mathcal{M}(p, r; f) \times \overline{\mathcal{M}}(r, q, f; H)\\ &\cup \bigcup_{m_f(r)=k+1} \overline{\mathcal{M}}(p, r, f; H) \times \mathcal{M}(r, q; f)\\ &\cup \bigcup_{\mu_N(x)+\dim N/2=k} \mathcal{M}(p, f, g; x, H, J) \times \mathcal{M}(x, H, J; q, f, g).\end{aligned}$$

PROOF. Consider a sequence $(R_n, \gamma_-^n, \gamma_+^n, u_n)$ in $\overline{\mathcal{M}}(p, q, f; H)$. It either $W^{1,2}$-converges to an element of the same moduli space or one of the following four statements holds:

(1) There is a subsequence such that $R_{n_k} \to R_0$ and $(\gamma_-^{n_k}, \gamma_+^{n_k}, u_{n_k})$ converges to $(\gamma_-, \gamma_+, u) \in \mathcal{M}_{R_0}(p, q, f; H)$.
(2) There is a subsequence of $(R_n, \gamma_-^n, \gamma_+^n, u_n)$ that converges to a broken trajectory in $\mathcal{M}(p, r; f) \times \overline{\mathcal{M}}(r, q, f; H)$. The subsequence $(\gamma_+^{n_k}, u_{n_k})$ converges in $W^{1,2}$ topology and $\gamma_-^{n_k}$ converges weakly.
(3) There is a subsequence that converges to a broken trajectory in $\overline{\mathcal{M}}(p, r, f; H) \times \mathcal{M}(r, q; f)$, similarly to (2).

(4) There is a subsequence such that $R_{n_k} \to +\infty$ and $(\gamma_-^{n_k}, \gamma_+^{n_k}, u_{n_k})$ converges weakly to a broken element of $\mathcal{M}(p, f, g; x, H, J) \times \mathcal{M}(x, H, J; q, f, g)$.

If R_n is bounded, then we can find a compact K such that $\{R_n\} \subset K$. The family ρ_R can be chosen to depend continuously on R, so all estimates in Proposition 2.2 hold uniformly on $R \in K$. In a similar way to Proposition 2.2, we conclude that $(\gamma_-^n, \gamma_+^n, u_n)$ has a subsequence that converges locally uniformly. So, if $(R_n, \gamma_-^n, \gamma_+^n, u_n)$ does not converge to an element of $\overline{\mathcal{M}}(p, q, f; H)$, then $R_n \to R_0$ or $R_n \to R > R_0$ (R_n denotes the subsequence, as well). If the first case, $(\gamma_-^n, \gamma_+^n, u_n)$ converges in $W^{1,2}$ topology, and in the second one $(\gamma_-^n, \gamma_+^n, u_n)$ converges to a broken trajectory. Since the dimension of $\overline{\mathcal{M}}(p, q, f; H)$ is 1, it can break only once. The breaking can happen on trajectories γ_-^n or γ_+^n and not on the disc. The sequence u_n cannot converge to a broken disc because the nonholomorphic part of the domain is compact and u_n converges there. If it breaks on the holomorphic part, then we obtain a solution of a system

$$\begin{aligned}
&v : \mathbb{R} \times [0, 1] \to T^*M, \quad \frac{\partial v}{\partial s} + J\frac{\partial v}{\partial t} = 0,\\
&v(\mathbb{R} \times \{0\}) \subset o_M, \quad v(\mathbb{R} \times \{1\}) \subset \nu^*N.
\end{aligned}$$

We have already seen that all such solutions are constant, so u_n cannot break on the holomorphic part either. In this way, we covered the first three cases. The fourth case arises if the sequence R_n is not bounded. We can find a subsequence $R_n \to +\infty$. Then the discs

$$u_n^-(s, t) := u_n(s - R_n - R_0 - 1, t), \quad u_n^+(s, t) := u_n(s + R_n + R_0 + 1, t),$$

converge locally uniformly with all derivatives to some u^- and u^+, respectively. These discs are solutions of the system

$$\begin{aligned}
&\frac{\partial u^\pm}{\partial s} + J\Big(\frac{\partial u^\pm}{\partial t} - X_{\rho^\pm_{R_0}}(u^\pm)\Big) = 0,\\
&u^\pm(\mathbb{R} \times \{0\}) \subset o_M, \quad u^\pm(\mathbb{R} \times \{1\}) \subset \nu^*N,\\
&u^\pm(\mp\infty, t) = x(t), \quad u^\pm(\pm\infty, t) = \gamma_\pm(0).
\end{aligned}$$

The sequences $\gamma_\pm^n$ cannot break because of dimensional reason, so they converge to some trajectories $\gamma_\pm$.

Conversely, for each broken trajectory of some of the types

$$\begin{aligned}
&(\gamma, \gamma_-, \gamma_+, u) \in \mathcal{M}(p, r; f) \times \overline{\mathcal{M}}(r, q, f; H),\\
&(\gamma_-, \gamma_+, u, \gamma) \in \overline{\mathcal{M}}(p, r, f; H) \times \mathcal{M}(r, q; f),\\
&(\gamma_1, u_1, \gamma_2, u_2) \in \mathcal{M}(p, f, g; x, H, J) \times \mathcal{M}(x, H, J; q, f, g),
\end{aligned}$$

there is a sequence in $\overline{\mathcal{M}}(p, q, f; H)$ that converges weakly to a corresponding broken trajectory. The proof is based on the implicit-function theorem and pregluing and gluing techniques. □

We continue with the construction of the auxiliary manifold, again with the variable domain, that now connects the Hamiltonian orbits. Fix an $\varepsilon > 0$. Consider

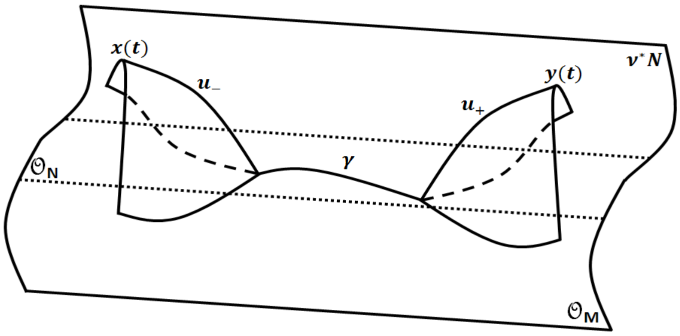

FIGURE 7. $\mathcal{M}_\varepsilon(x, y, H; f)$

the moduli space $\mathcal{M}_\varepsilon(x, y, H; f)$ defined as the set of maps

$$u_\pm : \mathbb{R} \times [0, 1] \to T^*M, \quad \gamma : [-\varepsilon, \varepsilon] \to N$$

that satisfy

$$\begin{aligned}
&\frac{\partial u_\pm}{\partial s} + J\Big(\frac{\partial u_\pm}{\partial t} - X_{\rho_R^\pm H}(u_\pm)\Big) = 0, \quad \frac{d\gamma}{ds} = -\nabla f(\gamma),\\
&E(u_\pm) < +\infty, \quad u_\pm(s, 0) \in o_M, \ u_\pm(s, 1) \in \nu^* N, \ s \in \mathbb{R},\\
&u_-(-\infty, t) = x(t), \ u_+(+\infty, t) = y(t), \quad u_\mp(\pm\infty) = \gamma(\mp\varepsilon),
\end{aligned}$$

(see Figure 7) and consider the moduli space

$$\underline{\mathcal{M}}(x, y, H; f) = \big\{(\varepsilon, u_-, u_+, \gamma) \mid (u_-, u_+, \gamma) \in \mathcal{M}_\varepsilon(x, y, H; f), \ \varepsilon \in [\varepsilon_0, \varepsilon_1]\big\},$$

where ε_0 and ε_1 are fixed positive numbers.

For $\mu_N(y) = \mu_N(x) + 1$, $\underline{\mathcal{M}}(x, y, H; f)$ is a zero-dimensional manifold. If $\mu_N(y) = \mu_N(x)$, then $\underline{\mathcal{M}}(x, y, H; f)$ is a one-dimensional manifold and we can describe its boundary.

PROPOSITION 2.5. *Let $x, y \in \mathrm{CF}_k(H)$. Then the topological boundary of $\underline{\mathcal{M}}(x, y, H; f)$ can be identified with*

$$\begin{aligned}
\partial\underline{\mathcal{M}}(x, y, H; f) = \ &\mathcal{M}_{\varepsilon_1}(x, y, H; f) \cup \mathcal{M}_{\varepsilon_0}(x, y, H; f)\\
&\cup \bigcup_{\mu_N(z) = \mu_N(x) - 1} \mathcal{M}(x, z; H) \times \underline{\mathcal{M}}(z, y, H; f)\\
&\cup \bigcup_{\mu_N(z) = \mu_N(x) + 1} \underline{\mathcal{M}}(x, z, H; f) \times \mathcal{M}(z, y; H).
\end{aligned}$$

PROOF. Let us take a sequence $(\varepsilon_n, u_-^n, u_+^n, \gamma_n) \in \underline{\mathcal{M}}(x, y, H; f)$ that has no convergent subsequence in $W^{1,2}$-topology. Since a sequence ε_n is bounded, all uniform estimates for $u_\pm^n, \gamma_n$ hold uniformly on ε (see Proposition 2.2). Hence, the sequences u_-^n, u_+^n and γ_n converge locally uniformly and (u_-^n, u_+^n, γ_n) can break only once (for dimensional reason). The domain of γ_n is bounded, so the trajectory γ_n cannot break. The only remaining possibilities are:

(1) There is a subsequence which converges to an element of $\mathcal{M}_{\varepsilon_1}(x, y, H; f)$ or $\mathcal{M}_{\varepsilon_0}(x, y, H; f)$.
(2) There is a subsequence which converges weakly to an element of $\mathcal{M}(x, z; H) \times \underline{\mathcal{M}}(z, y, H; f)$.
(3) There is a subsequence which converges weakly to an element of $\underline{\mathcal{M}}(x, z, H; f) \times \mathcal{M}(z, y; H)$. □

Now, we define moduli space similar to $\overline{\mathcal{M}}(p, q, f; H)$, except that we are not using a fixed Hamiltonian H, but a homotopy of Hamiltonians H_δ, $0 \leqslant \delta \leqslant 1$, that connects the given Hamiltonians H_0 and H_1,

$$\overline{\mathcal{M}}(p, q, f; H_\delta) = \big\{(\delta, \gamma_-, \gamma_+, u) \,|\, (\gamma_-, \gamma_+, u) \in \mathcal{M}_{R_0}(p, q, f; H_\delta)),\ 0 \leqslant \delta \leqslant 1\big\}.$$

The dimension of this manifold is $m_f(p) - m_f(q) + 1$, and its boundary is described in the following proposition.

Proposition 2.6. *Let $p, q \in \mathrm{CM}_k(f)$. Then the topological boundary of the one-dimensional manifold $\overline{\mathcal{M}}(p, q, f; H_\delta)$ can be identified with*

$$\begin{aligned}\partial\overline{\mathcal{M}}(p, q, f; H_\delta) = {} & \mathcal{M}_{R_0}(p, q, f; H_0) \cup \mathcal{M}_{R_0}(p, q, f; H_1)\\ & \cup \bigcup_{m_f(r)=k-1} \mathcal{M}(p, r; f) \times \overline{\mathcal{M}}(r, q, f; H_\delta)\\ & \cup \bigcup_{m_f(r)=k+1} \overline{\mathcal{M}}(p, r, f; H_\delta) \times \mathcal{M}(r, q; f).\end{aligned}$$

Proof. The proof is essentially the same as for Proposition 2.4. □

So far, we have discussed moduli spaces defined by a family of Hamiltonians with a fixed Morse function f. It will be useful to consider moduli spaces similar to $\mathcal{M}(p, f, g; x, H, J)$, that depend on a family of Morse functions and a family of Hamiltonians. Let $(f^{\alpha\beta}_{s,\delta}, H^{\alpha\beta}_{s,\delta})$, $0 \leqslant \delta \leqslant 1$, be a homotopy connecting $(f^\alpha, H^{\alpha\beta}_s)$ for $\delta = 0$ and $(f^{\alpha\beta}_s, H^\beta)$ for $\delta = 1$. Here

$$f^{\alpha\beta}_s = \begin{cases} f^\alpha, & s \leqslant -T-1\\ f^\beta, & s \geqslant -T\end{cases} \quad \text{and} \quad H^{\alpha\beta}_s = \begin{cases} H^\alpha, & s \leqslant T\\ H^\beta, & s \geqslant T+1\end{cases}$$

are homotopies connecting the Morse functions f^α, f^β, and the Hamiltonians H^α, H^β, respectively

We choose a homotopy $(f^{\alpha\beta}_{s,\delta}, H^{\alpha\beta}_{s,\delta})$ such that for any δ and s negative (positive) enough, $f^{\alpha\beta}_{s,\delta}$ is equal to f^α ($H^{\alpha\beta}_{s,\delta}$ is equal to H^β). In the same way we choose a homotopy $(g^{\alpha\beta}_{s,\delta}, J^{\alpha\beta}_{s,\delta})$. Let $\widehat{\mathcal{M}}(p^\alpha, f^{\alpha\beta}_{s,\delta}; x^\beta, H^{\alpha\beta}_{s,\delta})$ be the set of the triples (δ, γ, u) such that

$$\begin{aligned}
&\gamma:(-\infty,0]\to N,\quad u:\mathbb{R}\times[0,1]\to T^*M,\quad \frac{d\gamma}{ds}=-\nabla_{g^{\alpha\beta}_{s,\delta}}f^{\alpha\beta}_{s,\delta}(\gamma(s)),\\
&\frac{\partial u}{\partial s}+J^{\alpha\beta}_{s,\delta}\Big(\frac{\partial u}{\partial t}-X_{\rho_R^+H^{\alpha\beta}_{s,\delta}}(u)\Big)=0,\quad E(u)<+\infty,\quad \gamma(-\infty)=p^\alpha,\\
&u(s,0)\in o_M,\ u(s,1)\in\nu^*N,\ s\in\mathbb{R},\quad u(+\infty,t)=x^\beta(t),\quad \gamma(0)=u(-\infty).
\end{aligned}\tag{2.3}$$

The dimension of this manifold is $m_{f^\alpha}(p^\alpha)-(\mu_N(x^\beta)+\frac{1}{2}\dim N)+1$. We also define the moduli space $\mathcal{M}(p^\alpha,f^{\alpha\beta}_s;x^\beta,H^\beta)$ as the set of pairs (γ,u) that satisfy

$$\begin{aligned}
&\gamma:(-\infty,0]\to N,\quad u:\mathbb{R}\times[0,1]\to T^*M,\quad \frac{d\gamma}{ds}=-\nabla_{g^{\alpha\beta}_s}f^{\alpha\beta}_s(\gamma(s)),\\
&\frac{\partial u}{\partial s}+J^\beta\Big(\frac{\partial u}{\partial t}-X_{\rho_R^+H^\beta}(u)\Big)=0,\quad E(u)<+\infty,\quad \gamma(-\infty)=p^\alpha,\\
&u(s,0)\in o_M,\ u(s,1)\in\nu^*N,\ s\in\mathbb{R},\quad u(+\infty,t)=x^\beta(t),\quad \gamma(0)=u(-\infty).
\end{aligned}$$

Let $\mathcal{M}(p^\alpha,f^\alpha;x^\beta,H^{\alpha\beta}_s)$ be the set of maps $\gamma:(-\infty,0]\to N,\ u:\mathbb{R}\times[0,1]\to T^*M$ such that

$$\begin{aligned}
&\frac{d\gamma}{ds}=-\nabla_{g^\alpha}f^\alpha(\gamma),\quad \frac{\partial u}{\partial s}+J^{\alpha\beta}_s\Big(\frac{\partial u}{\partial t}-X_{\rho_R^+H^{\alpha\beta}_s}(u)\Big)=0,\\
&E(u)<+\infty,\quad \gamma(-\infty)=p^\alpha,\quad \gamma(0)=u(-\infty)\\
&u(s,0)\in o_M,\ u(s,1)\in\nu^*N,\ s\in\mathbb{R},\quad u(+\infty,t)=x^\beta(t).
\end{aligned}\tag{2.4}$$

The manifolds $\mathcal{M}(p^\alpha,f^{\alpha\beta}_s;x^\beta,H^\beta)$ and $\mathcal{M}(p^\alpha,f^\alpha;x^\beta,H^{\alpha\beta}_s)$ are the two components of a boundary $\partial\widehat{\mathcal{M}}(p^\alpha,f^{\alpha\beta}_{s,\delta};x^\beta,H^{\alpha\beta}_{s,\delta})$ which we completely describe in the next proposition.

Proposition 2.7. *Let $m_{f^\alpha}(p^\alpha)=\mu_N(x^\beta)+\frac{1}{2}\dim N$. Then the topological boundary of one-dimensional manifold $\widehat{\mathcal{M}}(p^\alpha,f^{\alpha\beta}_{s,\delta};x^\beta,H^{\alpha\beta}_{s,\delta})$ can be identified with*

$$\begin{aligned}
\partial\widehat{\mathcal{M}}(p^\alpha,f^{\alpha\beta}_{s,\delta};x^\beta,H^{\alpha\beta}_{s,\delta})&=\mathcal{M}(p^\alpha,f^{\alpha\beta}_s;x^\beta,H^\beta)\cup\mathcal{M}(p^\alpha,f^\alpha;x^\beta,H^{\alpha\beta}_s)\\
&\cup\bigcup_{m_{f^\alpha}(q^\alpha)=m_{f^\alpha}(p^\alpha)-1}\mathcal{M}(p^\alpha,q^\alpha;f^\alpha)\times\widehat{\mathcal{M}}(q^\alpha,f^{\alpha\beta}_{s,\delta};x^\beta,H^{\alpha\beta}_{s,\delta})\\
&\cup\bigcup_{\mu_N(y^\beta)=\mu_N(x^\beta)+1}\widehat{\mathcal{M}}(p^\alpha,f^{\alpha\beta}_{s,\delta};y^\beta,H^{\alpha\beta}_{s,\delta})\times\mathcal{M}(y^\beta,x^\beta;H^\beta).
\end{aligned}$$

Proof. The proof is essentially the same as for Proposition 2.4. □

3. Isomorphism

We saw in Propositions 2.2 and 2.3 that $\mathcal{M}(p,f,g;x,H,J)$ and $\mathcal{M}(x,H,J;p,f,g)$ are finite sets if $m_f(p)=\mu_N(x)+\frac{1}{2}\dim N$. Denote their cardinal numbers (modulo 2) by $n(p,f,g;x,H,J)$ and $n(x,H,J;p,f,g)$ and define homomorphisms on generators:

$$\phi:\mathrm{CF}_k(H)\to\mathrm{CM}_k(f),\quad \phi(x)=\sum_{m_f(p)=k}n(x,H,J;p,f,g)\,p,$$

$$\psi:\mathrm{CM}_k(f)\to\mathrm{CF}_k(H),\quad \psi(p)=\sum_{\mu_N(x)=k-\frac{1}{2}\dim N}n(p,f,g;x,H,J)\,x.$$

PROPOSITION 3.1. *The homomorphisms ϕ and ψ are well defined chain maps.*

PROOF. It follows from Propositions 2.2, 2.3 and from the way in which the chain complexes $\mathrm{CM}_*(f)$ and $\mathrm{CF}_*(H)$ are graded that these homomorphisms are well defined. We prove that $(\phi\circ\partial_F-\partial_M\circ\phi)(x)=0$ for all $x\in\mathrm{CF}_k(H)$. We have

$$\begin{aligned}(\phi\circ\partial_F-\partial_M\circ\phi)(x)&\\ &=\sum_{m_f(q)=k-1}\Bigg(\sum_{\mu_N(y)+\dim N/2=k-1} n(x,y;H)\,n(y,H,J;q,f,g)\Bigg)q\\ &\quad-\sum_{m_f(q)=k-1}\Bigg(\sum_{m_f(p)=k} n(x,H,J;p,f,g)\,n(p,q;f)\Bigg)q.\end{aligned}$$

Let $p\in\mathrm{CM}_k(f)$, $q\in\mathrm{CM}_{k-1}(f)$ and $y\in\mathrm{CF}_{k-1}(H)$. From Proposition 2.3 it follows

$$\begin{aligned}\sum_{\mu_N(y)+\dim N/2=k-1} & n(x,y;H)\,n(y,H,J;q,f,g)\\ &-\sum_{m_f(p)=k} n(x,H,J;p,f,g)\,n(p,q;f)=0,\end{aligned}$$

since it is the number (modulo 2) of ends of the one-dimensional manifold

$$\mathcal{M}(x,H,J;q,f,g).$$

So, $(\phi\circ\partial_F-\partial_M\circ\phi)(x)=0$.

The proof of the identity $\psi\circ\partial_M=\partial_F\circ\psi$ is analogous. $\square$

From the previous proposition, it follows that ϕ and ψ induce homomorphisms

$$\Phi:\mathrm{HF}_k(H)\to\mathrm{HM}_k(f),\quad \Psi:\mathrm{HM}_k(f)\to\mathrm{HF}_k(H).$$

in homology. They are PSS-type isomorphisms. Now, we can prove Theorem 1.1. From the fact that these homomorphisms are inverse to each other, it immediately follows that they are isomorphisms. In order to show this, we prove that $\phi\circ\psi$ and $\psi\circ\phi$ are maps, chain homotopic to the identity.

PROOF OF THEOREM 1.1. If we look at a composition of homomorphisms ϕ and ψ,

$$\phi\circ\psi(p)=\sum_{m_f(q)=k}\Bigg(\sum_{\mu_N(x)+\dim N/2=k} n(p,f,g;x,H,J)\,n(x,H,J;q,f,g)\Bigg)q,$$

we can see that $\sum_x n(p,f,g;x,H,J)n(x,H,J;q,f,g)$ is the number of points of the set $\bigcup_x\mathcal{M}(p,f,g;x,H,J)\times\mathcal{M}(x,H,J;q,f,g)$, which is a component of the boundary $\partial\overline{\mathcal{M}}(p,q,f;H)$.

Similarly to [**15**] we define homomorphisms l and j,

$$l:\mathrm{CM}_k(f)\to\mathrm{CM}_k(f),\quad l(p)=\sum_{m_f(q)=k} n(p,q,f;H)\,q,$$

$$j : \mathrm{CM}_k(f) \to \mathrm{CM}_{k+1}(f), \quad j(p) = \sum_{m_f(r)=k+1} \overline{n}(p, r, f; H)\, r.$$

Here $n(p, q, f; H)$ is the number of intersections of the space of perturbed holomorphic discs with the unstable manifold $W^u(p, f)$ and the stable manifold $W^s(q, f)$. We consider discs with one half of the boundary on the zero-section o_M, and the other half on the conormal bundle ν^*N. In other words, $n(p, q, f; H)$ is the number of elements of $\mathcal{M}_{R_0}(p, q, f; H)$. By $\overline{n}(p, r, f; H)$ we denote the number of elements of a zero-dimensional manifold $\overline{\mathcal{M}}(p, r, f; H)$. The sum

$$\sum_{m_f(r)=k-1} n(p, r; f)\, \overline{n}(r, q, f; H)$$

corresponds to the sum that occurs in $j \circ \partial_M$, and

$$\sum_{m_f(r)=k+1} \overline{n}(p, r, f; H)\, n(r, q; f)$$

corresponds to the sum in $\partial_M \circ j$. From Proposition 2.4, it follows

$$\phi \circ \psi - l = \partial_M \circ j + j \circ \partial_M.$$

Now, we prove that homomorphism $L : \mathrm{HM}_k(f) \to \mathrm{HM}_k(f)$, in homology, induced by the chain homomorphism l, does not depend on the Hamiltonian H. Let H_0 and H_1 be Hamiltonians and H_δ, $0 \leqslant \delta \leqslant 1$, a homotopy between them; l_0 and l_1 are chain homomorphisms corresponding to H_0 and H_1. From Proposition 2.6 we get the relation $l_1 - l_0 = \partial_M \circ j_\delta + j_\delta \circ \partial_M$, where

$$j_\delta : \mathrm{CM}_k(f) \to \mathrm{CM}_{k+1}(f), \qquad j_\delta(p) = \sum_{m_f(r)=k+1} \overline{n}(p, r, f; H_\delta)\, r.$$

Here, $\overline{n}(p, r, f; H_\delta)$ is the number of elements of $\overline{\mathcal{M}}(p, r, f; H_\delta)$. If we choose a homotopy between our Hamiltonian H and 0, then we conclude that the map l is chain homotopic to the map $i : \mathrm{CM}_k(f) \to \mathrm{CM}_k(f)$, $i(p) = \sum_{m_f(q)=k} n(p, q, f; 0)\, q$. Thus, L and the map I, induced by i, are the same maps in homology. We explained above that unperturbed holomorphic disc, with one half of the boundary on the zero-section and the other half on the conormal bundle, is constant. It follows that $n(p, q, f; 0)$ is the number of points in $W^u(p, f) \cap W^s(q, f)$. Considering Morse indices of p and q, we get $I = \mathbb{I}$.

We use the same idea to prove $\Psi \circ \Phi = \mathbb{I}$. The composition $\psi \circ \phi$ is chain homotopic to a chain homomorphism $r : \mathrm{CF}_k(H) \to \mathrm{CF}_k(H)$ which induces the identity in homology. If we denote by $n_\varepsilon(x, y, H; f)$ the number of elements of the zero-dimensional manifold $\mathcal{M}_\varepsilon(x, y, H; f)$, then the map analogous to l is

$$r(x) = \sum_{\mu_N(y)=\mu_N(x)} n_\varepsilon(x, y, H; f)\, y.$$

Similarly to the first part of the proof, a homomorphism in homology induced by r is independent of the choice of ε. Let r_0 and r_1 be homomorphisms corresponding

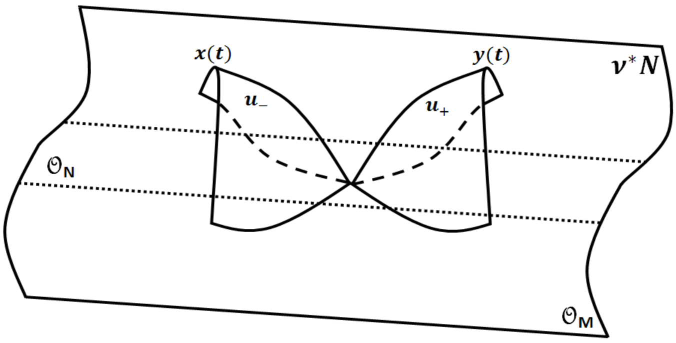

FIGURE 8. $\widetilde{\mathcal{M}}(x, y; H)$

to the values ε_0 and ε_1. We define a chain homomorphism

$$s : \mathrm{CF}_k(H) \to \mathrm{CF}_{k+1}(H), \qquad s(x) = \sum_{\mu_N(y)+\dim N/2=k+1} \underline{n}(x, y, H; f)\, y,$$

where $\underline{n}(x, y, H; f)$ denotes the number of elements of $\underline{\mathcal{M}}(x, y, H; f)$. From Proposition 2.5 we conclude that $r_0 - r_1 = s \circ \partial_F + \partial_F \circ s$. If we pass to the limit as $\varepsilon \to 0$ we get that $\psi \circ \phi$ if chain homotopic to the homomorphism

$$\widetilde{i} : \mathrm{CF}_k(H) \to \mathrm{CF}_k(H), \quad \widetilde{i}(x) = \sum_{\mu_N(y)+\dim N/2=k} \widetilde{n}(x, y; H)\, y.$$

Here $\widetilde{n}(x, y; H)$ is the number of elements of the zero-dimensional manifold $\widetilde{\mathcal{M}}(x, y; H)$ defined as the set of pairs (u_-, u_+) such that (see Figure 8)

$$\begin{aligned}
&u_\pm : \mathbb{R} \times [0, 1] \to T^*M, \;\; E(u_\pm) < \infty,\\
&\frac{\partial u_\pm}{\partial s} + J\Big(\frac{\partial u_\pm}{\partial t} - X_{\rho_R^\pm H}(u_\pm)\Big) = 0,\\
&u_\pm(s, 0) \in o_M,\; u_\pm(s, 1) \in \nu^*N,\; s \in \mathbb{R},\\
&u_-(-\infty, t) = x(t),\; u_+(+\infty, t) = y(t),\\
&u_-(+\infty) = u_+(-\infty).
\end{aligned}$$

In the rest of the proof, we show that counting the number of elements of $\widetilde{\mathcal{M}}(x, y; H)$ is the same as counting the pseudo holomorphic strips between x and y (at the homology level). The main idea is to show that $\widetilde{\mathcal{M}}$ is cobordant to the manifold that consists of appropriate pseudo holomorphic strips.

We define auxiliary manifold $\mathcal{M}_R(x, y; H)$ as the set of all maps $u : \mathbb{R} \times [0, 1] \to T^*M$ such that

$$\begin{aligned}
&\frac{\partial u}{\partial s} + J\Big(\frac{\partial u}{\partial t} - X_{\rho_R H}(u)\Big) = 0,\\
&u(s, 0) \in o_M,\; u(s, 1) \in \nu^*N,\; s \in \mathbb{R},\\
&u(-\infty, t) = x(t),\; u(+\infty, t) = y(t).
\end{aligned}$$

Its parameterised version is

$$\check{\mathcal{M}}(x, y; H) = \{(R, u) \mid R \geqslant R_0,\, u \in \mathcal{M}_R(x, y; H)\}\,.$$

Here, $\rho_R : \mathbb{R} \to \mathbb{R}$ is a smooth function such that

$$\rho_R(s) = \begin{cases} 1, & |s| \geqslant R + 1 \\ 0, & |s| \leqslant R. \end{cases}$$

The boundary of the manifold $\check{\mathcal{M}}(x, y; H)$ can be identified with

$$\begin{aligned} \partial \check{\mathcal{M}} = \mathcal{M}_{R_0}(x, y; H) \ &\cup \ \widetilde{\mathcal{M}}(x, y; H) \\ &\cup \ \bigcup_z \mathcal{M}(x, z; H) \times \check{\mathcal{M}}(z, y; H) \\ &\cup \ \bigcup_z \check{\mathcal{M}}(x, z; H) \times \mathcal{M}(z, y; H). \end{aligned} \tag{3.1}$$

Now we explain equality (3.1). It is clear how the last two terms on the right-hand side appear at the boundary. The elements from $\mathcal{M}_{R_0}(x, y; H)$ appear at the boundary when $R_n \to R_0$. The most complicated part is to prove that $\widetilde{\mathcal{M}}$ is a part of the boundary. We show that in two steps. In Step A we explain why it holds $\partial \check{\mathcal{M}} \subset \widetilde{\mathcal{M}}$. And in Step B we show the opposite inclusion, $\widetilde{\mathcal{M}} \subset \partial \check{\mathcal{M}}$.

Step A. When $R_n \to +\infty$, we can identify the limit of $u_n \in \mathcal{M}_{R_n}(x, y; H)$ with the element from $\widetilde{\mathcal{M}}(x, y; H)$ using the reparameterisation

$$u_n^-(s, t) = u_n(s - R_n + R_0, t), \ \ u_n^+(s, t) = u_n(s + R_n - R_0, t).$$

The strip u_n^- satisfies the equation

$$\frac{\partial u_n^-}{\partial s} + J\left(\frac{\partial u_n^-}{\partial t} - X_{\rho_{R_n-}H}(u_n^-)\right) = 0,$$

and the boundary conditions

$$\begin{aligned} u_n^-(s, 0) &= u_n(s - R_n + R_0, 0) \in o_M, \\ u_n^-(s, 1) &= u_n(s - R_n + R_0, 1) \in \nu^* N, \end{aligned}$$

for $s \in \mathbb{R}$. The function ρ_{R_n-} is

$$\rho_{R_n-}(s) = \begin{cases} 0, & -R_0 \leqslant s \leqslant 2R_n - R_0 \\ 1, & s \in (-\infty, -R_0 - 1] \cup [2R_n - R_0 + 1, +\infty). \end{cases}$$

The positive strip u_n^+ also satisfies the perturbed Cauchy–Riemann equation

$$\frac{\partial u_n^+}{\partial s} + J\left(\frac{\partial u_n^+}{\partial t} - X_{\rho_{R_n+}H}(u_n^+)\right) = 0,$$

the line $u_n^+(\mathbb{R} \times \{0\})$ is on the zero section and $u_n^+(\mathbb{R} \times \{1\})$ is on the conormal bundle. The function ρ_{R_n+} is defined by

$$\rho_{R_n+}(s) = \begin{cases} 0, & -2R_n + R_0 \leqslant s \leqslant R_0 \\ 1, & s \in (-\infty, -2R_n + R_0 - 1] \cup [R_0 + 1, +\infty). \end{cases}$$

The strip $u_n^{\pm}$ converges locally uniformly with all derivatives to some $u^{\pm}$ that satisfies the equation

$$\frac{\partial u^{\pm}}{\partial s} + J\Big(\frac{\partial u^{\pm}}{\partial t} - X_{\rho^{\pm}_{R_0} H}(u^{\pm})\Big) = 0.$$

It is obvious that $u^-(-\infty, t) = x(t)$ and $u^+(+\infty, t) = y(t)$. At the $+\infty$-end, the strip u^- converges to a point $p \in N \subset o_M$ since u^- is holomorphic for $s \geqslant -R_0$ and it has a finite energy. The positive strip u^+ is holomorphic at $-\infty$ and it converges to a point $q \in N$. Since $u_n^-(R_n - R_0, t) = u_n^+(-R_n + R_0, t)$, we conclude that $p = q$. Thus, the pair (u^-, u^+) belongs to $\widetilde{\mathcal{M}}(x, y; H)$.

Step B. For a given $(u_-, u_+) \in \widetilde{\mathcal{M}}$, we can find a sequence of elements $(R, \omega_R) \in \check{\mathcal{M}}$ that Gromov-converges to (u_-, u_+) as $R \to +\infty$ (see [**4**, Theorem 4.1.2], [**9**, Chapter 4.7] and [**31**, Theorem 7.1]).

The main technique is gluing, and goes as follows. The strips u_- and u_+ are holomorphic around the point $u_-(+\infty) = u_+(-\infty)$, and we can preglue them to obtain a map u_R. This is an approximate solution of the Cauchy–Riemann equation, u_R satisfies it everywhere except a small neighbourhood of $u_R(0) = u_-(+\infty) = u_+(-\infty)$. Next, we construct a right inverse to the linearization D_{u_R} of the operator $\overline{\partial}$. Using the implicit-function theorem, we find a genuine solution ω_R to this equation, that is in a neighborhood of an approximate solution.

Biran and Cornea in [**4**] glued two holomorphic discs with the boundary on one Lagrangian submanifold. Frauenfelder in [**9**] and Schmäschke in [**31**] worked with two cleanly intersecting (compact) submanifolds in a compact symplectic manifold. The cotangent bundle is not a compact manifold, but, with appropriate choice on an almost complex structure (see the definition of j^c below), the image of every holomorphic strip lies in a compact subset of T^*M [**23**, Theorem 3.2]. So we can assume that everything happens in a compact subset of our symplectic manifold. We also need a special Riemannian metrics on T^*M such that o_M and ν^*N are totally geodesic submanifolds with respect to these metrics.

Following [**24, 10, 33**], we explain choices on almost complex structures and Riemannian metrics on T^*M. Fix a Riemannian metric g on M. The associated Levi-Civita connection induces the canonical almost complex structure on T^*M, which we denote by J_g. We define the subset j^c of the set of almost complex structure on T^*M by

$$j^c = \{J \mid J \text{ is compatible to } \omega,\ J = J_g \text{ outside a compact subset in } T^*M\}.$$

Let J_t be a smooth path in j^c. Then there exists a smooth family of metrics g_t such that

(1) o_M is totally geodesic with respect to g_0 and $J_0(q)T_q o_M$ is the orthogonal complement of $T_q o_M$ for every $q \in o_M$,
(2) ν^*N is totally geodesic with respect to g_1 and $J_1(q)T_q(\nu^*N)$ is the orthogonal complement of $T_q(\nu^*N)$ near the intersection point of two holomorphic strips that we glue,
(3) $g_t(J_t(q)u, J_t(q)v) = g_t(u, v)$ for $q \in T^*M$ and $u, v \in T_q(T^*M)$.

We can define a metric g_0 such that

(1) o_M is totally geodesic with respect to g_0 and $J_0(q)T_q o_M$ is the orthogonal complement of $T_q o_M$ for every $q \in o_M$,
(2) $g_0(J_0(q)u, J_0(q)v) = g_0(u, v)$ for $q \in T^*M$ and $u, v \in T_q(T^*M)$,

see [**10**] for details. In the same way we can define a metric g_1 that satisfies the same properties for the submanifold ν^*N. In [**10**] the author assumes that the Lagrangian submanifold is compact. The conormal bundle ν^*N is not a compact manifold, in general. But it is enough to find a metric such that ν^*N is a totally geodesic submanifold near $N \subset o_M$, not the whole ν^*N. The linear combination $g_t(u, v) = \overline{g}_t(u, v) + \overline{g}_t(J_t u, J_t v)$, where $\overline{g}_t(u, v) = (1-t)g_0(u, v) + tg_1(u, v)$, gives an appropriate family of metrics (see also [**33**]). All the other technical details of gluing are the same as in [**31**].

Now, we return to the homomorphism $\widetilde{i}$. Using the one-dimensional component of $\check{\mathcal{M}}(x, y; H)$ and the description of its boundary (3.1), we conclude that $\widetilde{i}$ (i.e., $\psi \circ \phi$) is chain homotopic to the map

$$k : x \mapsto \sum_{\mu_N(y)=\mu_N(x)} n(x, y; H)\, y.$$

If there is a nonconstant holomorphic strip that connects Hamiltonian orbits x and y, then $\mu_N(x) > \mu_N(y)$. It follows that

$$n(x, y; H) = \begin{cases} 1, & x = y \\ 0, & x \neq y, \end{cases}$$

i.e., the map k induces the identity in homology $\mathrm{HF}_*(H)$. □

4. Commutative diagram

Proof of Theorem 1.2. This theorem states that $S^{\alpha\beta} \circ \Psi^{\alpha} = \Psi^{\beta} \circ T^{\alpha\beta}$. The composition on the left-hand side is generated bay the map $\sigma^{\alpha\beta} \circ \psi^{\alpha}$, and the right-hand side is generated by $\psi^{\beta} \circ \tau^{\alpha\beta}$ on the chain level. The idea is to prove that these maps on the chain level are homotopic to each other.

We separate proof in two steps. In Step 1 and Step 2 we define new maps χ and ξ that are homotopic to $\sigma^{\alpha\beta} \circ \psi^{\alpha}$ and $\psi^{\beta} \circ \tau^{\alpha\beta}$, respectively. In the conclusion of the proof we show that χ and ξ are chain homotopic maps.

Step 1. From definitions it follows

$$(\sigma^{\alpha\beta} \circ \psi^{\alpha})(p^{\alpha}) = \sum_{x^{\alpha}, x^{\beta}} n(p^{\alpha}, f^{\alpha}, g^{\alpha}; x^{\alpha}, H^{\alpha}, J^{\alpha})\, n(x^{\alpha}, x^{\beta}; H^{\alpha\beta})\, x^{\beta}.$$

This means that $\sigma^{\alpha\beta} \circ \psi^{\alpha}$ counts the number of points of the set

$$\bigcup_{x^{\alpha}} \mathcal{M}(p^{\alpha}, f^{\alpha}, g^{\alpha}; x^{\alpha}, H^{\alpha}, J^{\alpha}) \times \mathcal{M}(x^{\alpha}, x^{\beta}; H^{\alpha\beta}),$$

where $\mathcal{M}(x^{\alpha}, x^{\beta}; H^{\alpha\beta})$ denotes the set of solutions of (1.2). Summation is taken over x^{α}, x^{β} such that $m_{f^{\alpha}}(p^{\alpha}) = \mu_N(x^{\alpha}) + \frac{1}{2}\dim N = \mu_N(x^{\beta}) + \frac{1}{2}\dim N$. Let us define a family of homotopies between Hamiltonians H^α and H^β by

$$H^{\alpha\beta}_{T,s} = \begin{cases} H^\alpha, & s \leqslant T \\ H^\beta, & s \geqslant T+1, \end{cases}$$

and a family of homotopies between almost complex structures J^α and J^β by

$$J^{\alpha\beta}_{T,s} = \begin{cases} J^\alpha, & s \leqslant T \\ J^\beta, & s \geqslant T+1. \end{cases}$$

We consider the moduli space $\breve{\mathcal{M}}(p^\alpha, f^\alpha; x^\beta, H^{\alpha\beta}_{T,s})$ defined as the set of the triples (T, γ, u) such that

$$\begin{aligned} &T \geqslant T_0, \quad \gamma : (-\infty, 0] \to N, \quad u : \mathbb{R} \times [0,1] \to T^*M, \\ &\frac{d\gamma}{ds} = -\nabla_{g^\alpha} f^\alpha(\gamma), \quad \frac{\partial u}{\partial s} + J^{\alpha\beta}_{T,s}\Big(\frac{\partial u}{\partial t} - X_{\rho_R^+ H^{\alpha\beta}_{T,s}}(u)\Big) = 0, \\ &E(u) < +\infty, \quad \gamma(-\infty) = p^\alpha, \quad u(s,0) \in o_M, \; u(s,1) \in \nu^*N, \; s \in \mathbb{R}, \\ &u(+\infty, t) = x^\beta(t), \quad \gamma(0) = u(-\infty). \end{aligned}$$

Using the same idea as in the proof of Theorem 1.1, from gluing and compactness arguments, it follows that boundary of $\breve{\mathcal{M}}$ can be described as

$$\begin{aligned} \partial\breve{\mathcal{M}}(p^\alpha, f^\alpha; x^\beta, H^{\alpha\beta}_{T,s}) = {} & \mathcal{M}(p^\alpha, f^\alpha; x^\beta, H^{\alpha\beta}_{T_0,s}) \\ & \cup \bigcup_{x^\alpha} \mathcal{M}(p^\alpha, f^\alpha, g^\alpha; x^\alpha, H^\alpha, J^\alpha) \times \mathcal{M}(x^\alpha, x^\beta; H^{\alpha\beta}) \\ & \cup \bigcup_{q^\alpha} \mathcal{M}(p^\alpha, q^\alpha; f^\alpha) \times \breve{\mathcal{M}}(q^\alpha, f^\alpha; x^\beta, H^{\alpha\beta}_{T,s}) \\ & \cup \bigcup_{y^\beta} \breve{\mathcal{M}}(p^\alpha, f^\alpha; y^\beta, H^{\alpha\beta}_{T,s}) \times \mathcal{M}(y^\beta, x^\beta; H^\beta). \end{aligned}$$

The first element in a previous union is already described in (2.4) (for fixed homotopy $H^{\alpha\beta}_s = H^{\alpha\beta}_{T_0,s}$). We define a map χ that counts the number of elements in $\mathcal{M}(p^\alpha, f^\alpha; x^\beta, H^{\alpha\beta}_s)$ by $\chi(p^\alpha) = \sum_{x^\beta} n(p^\alpha, f^\alpha; x^\beta, H^{\alpha\beta}_s)\, x^\beta$. From the description of topological boundary of $\breve{\mathcal{M}}$, we conclude that χ and $\sigma^{\alpha\beta} \circ \psi^\alpha$ are chain homotopic maps.

Step 2. The other composition satisfies the equation

$$\psi^\beta \circ \tau^{\alpha\beta}(p^\alpha) = \sum_{p^\beta, x^\beta} n(p^\alpha, p^\beta; f^{\alpha\beta})\, n(p^\beta, f^\beta, g^\beta; x^\beta, H^\beta, J^\beta)\, x^\beta.$$

Now, $\psi^\beta \circ \tau^{\alpha\beta}$ counts the number of points of the set

$$\bigcup_{p^\beta} \mathcal{M}(p^\alpha, p^\beta; f^{\alpha\beta}) \times \mathcal{M}(p^\beta, f^\beta, g^\beta; x^\beta, H^\beta, J^\beta),$$

where $\mathcal{M}(p^\alpha, p^\beta; f^{\alpha\beta})$ is the set of solutions of (1.4). Here, we take a sum over p^β, x^β such that $m_{f^\alpha}(p^\alpha) = m_{f^\beta}(p^\beta) = \mu_N(x^\beta) + \frac{1}{2}\dim N$. We define a map ξ

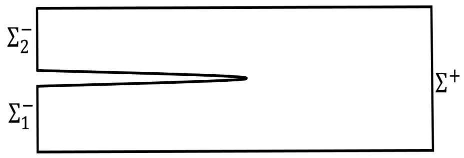

FIGURE 9. Riemannian surface Σ

that counts the number of points in $\mathcal{M}(p^\alpha, f_s^{\alpha\beta}; x^\beta, H^\beta)$. It follows, similarly as in Step 1, that ξ and $\psi^\beta \circ \tau^{\alpha\beta}$ are chain homotopic maps.

Using the moduli space $\widehat{\mathcal{M}}(p^\alpha, f_{s,\delta}^{\alpha\beta}; x^\beta, H_{s,\delta}^{\alpha\beta})$ defined in (2.3), we prove that χ and ξ are chain homotopic maps. Let us define a chain homomorphism

$$j : \mathrm{CM}_{k-1}(f^\alpha) \to \mathrm{CF}_k(H^\beta), \quad j(p^\alpha) = \sum_{\mu_N(x^\beta)+\dim N/2=k} \widehat{n}(p^\alpha, f_{s,\delta}^{\alpha\beta}; x^\beta, H_{s,\delta}^{\alpha\beta})\, x^\beta,$$

where $\widehat{n}(p^\alpha, f_{s,\delta}^{\alpha\beta}; x^\beta, H_{s,\delta}^{\alpha\beta})$ is the number of elements of the zero-dimensional manifold $\widehat{\mathcal{M}}(p^\alpha, f_{s,\delta}^{\alpha\beta}; x^\beta, H_{s,\delta}^{\alpha\beta})$. From Proposition 2.7, it follows that

$$\xi - \chi + j \circ \partial_M + \partial_F \circ j = 0. \qquad \square$$

5. Product in homology

In this section we prove Theorem 1.3. First we explain a construction of the product

$$\star : \mathrm{HF}_*(o_M, o_M : H_1) \otimes \mathrm{HF}_*(o_M, \nu^*N : H_2) \to \mathrm{HF}_*(o_M, \nu^*N : H_3).$$

Then we prove subadditivity of spectral invariants with respect to this product.

PROOF OF THEOREM 1.3. We define a Riemannian surface with boundary Σ as the disjoint union $\mathbb{R}\times[-1,0]\sqcup\mathbb{R}\times[0,1]$ with identification $(s,0^-)\sim(s,0^+)$ for $s\geqslant 0$ (see Figure 9). The surface Σ is conformally equivalent to a closed disc with three boundary punctures. Complex structure on $\Sigma\setminus\{(0,0)\}$ is induced by the inclusion $(s,t)\mapsto s+it$, in $\mathbb{C}$. Complex structure at $(0,0)$ is given by the square root.

Denote by Σ_1^-, Σ_2^-, Σ^+ the two “incoming" and one “outgoing" ends, such that

$$\Sigma_1^-, \Sigma_2^- \approx [0,1]\times(-\infty,0], \quad \Sigma^+\approx[0,1]\times[0,+\infty).$$

By $u_j^- := u|_{\Sigma_j^-}$, $j=1,2$, and $u^+ := u|_{\Sigma^+}$, we denote the restriction of the map defined on the surface Σ. Let $\rho^\pm : \mathbb{R}\to[0,1]$ denote the smooth cut-off functions such that

$$\rho^-(s) = \begin{cases} 1, & s\leqslant -2 \\ 0, & s\geqslant -1 \end{cases} \quad \rho^+(s) = \rho^-(-s).$$

For $x_1^- \in \mathrm{CF}_*(o_M, o_M : H_1)$, $x_2^- \in \mathrm{CF}_*(o_M, \nu^*N : H_2)$ and $x^+ \in \mathrm{CF}_*(o_M, \nu^*N : H_3)$, we define the moduli space $\mathcal{M}(x_1^-, x_2^-; x^+)$ as a set of maps $u : \Sigma \to T^*M$ such that

$$\partial_s u_j^- + J(\partial_t u_j^- - X_{\rho^- H_j}(u_j^-)) = 0, \ j = 1,2,$$

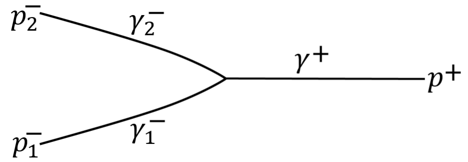

FIGURE 10. Set of trees $\mathcal{M}^{\mathrm{tree}}(p_1^-, p_2^-; p^+)$

$$\begin{aligned}
&\partial_s u^+ + J(\partial_t u^+ - X_{\rho^+ H_3}(u^+)) = 0,\\
&\partial_s u + J\partial_t u = 0 \ \text{ on } \ \Sigma_0 = \Sigma \smallsetminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3),\\
&u(s,-1) \in o_M,\ u(s,1) \in \nu^* N,\ s \in \mathbb{R},\\
&u(s,0^-),\ u(s,0^+) \in o_M,\ s \leqslant 0,\\
&u_j^-(-\infty,t) = x_j^-(t),\ j = 1,2,\ \ u^+(+\infty,t) = x^+(t).
\end{aligned}$$

We use the notation $\overline{\partial}_{J,H}(u) = 0$ for the perturbed Cauchy–Riemann equation that we consider in $\mathcal{M}(x_1^-, x_2^-; x^+)$. Elements of a moduli space $\mathcal{M}(x_1^-, x_2^-; x^+)$ are perturbed holomorphic discs u. The boundary of u is on the Lagrangian submanifold $o_M \cup \nu^* N$ with the clean self-intersection along N.

For generic choices of Hamiltonians and an almost complex structure, the manifold $\mathcal{M}(x_1^-, x_2^-; x^+)$ is smooth and of dimension

$$\mu_M(x_1^-) + \mu_N(x_2^-) - \mu_N(x^+) - \tfrac{1}{2}\dim M.$$

By $x_1^- \star x_2^- = \sum_{x^+} \sharp_2 \mathcal{M}(x_1^-, x_2^-; x^+)\, x^+$, we define a product on generators of Floer complexes. Here $\sharp_2\mathcal{M}(x_1^-, x_2^-; x^+)$ denotes the number (modulo 2) of elements of the zero-dimensional component of $\mathcal{M}(x_1^-, x_2^-; x^+)$. (Similar type of product is defined in [**6**], where it was used in the comparison of spectral invariants in Lagrangian and Hamiltonian Floer theory.) We extend the product $\star$ by bilinearity on $\mathrm{CF}_*(o_M, o_M : H_1) \otimes \mathrm{CF}_*(o_M, \nu^* N : H_2)$. The operation $\star$ commutes with the corresponding boundary operators and induces product in homology

$$\star : \mathrm{HF}_k(o_M, o_M : H_1) \otimes \mathrm{HF}_l(o_M, \nu^* N : H_2) \to \mathrm{HF}_{k+l-\dim M}(o_M, \nu^* N : H_3).$$

The next step is to define an exterior intersection product on the Morse homology. Let us take Morse functions $f_1 : M \to \mathbb{R}$ and $f_2, f_3 : N \to \mathbb{R}$. For $p_1^- \in \mathrm{Crit}(f_1)$, $p_2^- \in \mathrm{Crit}(f_2)$ and $p^+ \in \mathrm{Crit}(f_3)$ we define the set of trees $\mathcal{M}^{\mathrm{tree}}(p_1^-, p_2^-; p^+)$ (see Figure 10). The moduli space $\mathcal{M}^{\mathrm{tree}}(p_1^-, p_2^-; p^+)$ contains the triples $(\gamma_1^-, \gamma_2^-, \gamma^+)$ such that

$$\begin{gathered}
\gamma_1^- : (-\infty, 0] \to M,\ \ \gamma_2^- : (-\infty, 0] \to N,\ \ \gamma^+ : [0, +\infty) \to N,\\
\frac{d\gamma_1^-}{ds} = -\nabla_{g_1} f_1(\gamma_1^-),\ \ \frac{d\gamma_2^-}{ds} = -\nabla_{g_2} f_2(\gamma_2^-),\ \ \frac{d\gamma^+}{ds} = -\nabla_{g_3} f_3(\gamma+),\\
\gamma_1^-(-\infty) = p_1^-,\ \ \gamma_2^-(-\infty) = p_2^-,\ \ \gamma^+(+\infty) = p^+,\ \ \gamma_1^-(0) = \gamma_2^-(0) = \gamma^+(0).
\end{gathered}$$

For generic choices of Morse–Smale pairs (f_j, g_j), $j \in \{1,2,3\}$, $\mathcal{M}^{\mathrm{tree}}(p_1^-, p_2^-; p^+)$ is a smooth manifold of dimension $m_{f_1}(p_1^-) + m_{f_2}(p_2^-) - m_{f_3}(p^+) - \dim M$. On chain complexes we define

$$\cdot : \mathrm{CF}_k(M : f_1) \otimes \mathrm{CF}_l(N : f_2) \to \mathrm{CF}_{k+l-\dim M}(N : f_3),$$

by

$$p_1^- \cdot p_2^- = \sum_{p^+} \sharp_2 \mathcal{M}^{\mathrm{tree}}(p_1^-, p_2^-; p^+)\, p^+.$$

It is a chain map that defines the exterior intersection product

$$\cdot : \mathrm{HF}_k(M : f_1) \otimes \mathrm{HF}_l(N : f_2) \to \mathrm{HF}_{k+l-\dim M}(N : f_3).$$

Once we have defined the exterior intersection product, we can prove that PSS preserves the algebraic structure, i.e., it maps $\cdot$ to $\star$. The idea is to show that $\cdot$ and $\phi(\psi \star \psi)$ are chain homotopic maps. As in the previous situation, p_1^-, p_2^-, p^+ are critical points. We know that

$$\begin{aligned}\phi(\psi(p_1^-) \star \psi(p_2^-)) = \sum_{x_1^-, x_2^-, x^+, p^+} &\sharp_2 \mathcal{M}(p_1^-, f_1; x_1^-, H_1)\, \sharp_2 \mathcal{M}(p_2^-, f_2; x_2^-, H_2) \\ &\sharp_2 \mathcal{M}(x_1^-, x_2^-; x^+)\, \sharp_2 \mathcal{M}(x^+, H_3; p^+, f_3)\, p^+.\end{aligned}$$

Following [**16**], we define two auxiliary manifolds. The first of them, the manifold $\mathcal{M}_R^{\mathrm{prod}}(p_1^-, p_2^-, p^+; \vec{f}, \vec{H})$, is the set of all $(\gamma_1^-, \gamma_2^-, \gamma^+, u)$ that satisfy

$$\begin{aligned}&\gamma_1^- : (-\infty, 0] \to M,\ \gamma_1^-(-\infty) = p_1^-,\ \ \gamma_2^- : (-\infty, 0] \to N,\ \ \gamma_2^-(-\infty) = p_2^-,\\ &\gamma^+ : [0, +\infty) \to N,\ \ \gamma^+(+\infty) = p^+,\ \ u : \Sigma \to T^*M,\\ &\frac{d\gamma_1^-}{ds} = -\nabla_{g_1} f_1(\gamma_1^-),\ \ \frac{d\gamma_2^-}{ds} = -\nabla_{g_2} f_2(\gamma_2^-),\ \ \frac{d\gamma^+}{ds} = -\nabla_{g_3} f_3(\gamma+),\\ &\partial_s u_j^- + J(\partial_t u_j^- - X_{\kappa_R^- H_j}(u_j^-)) = 0,\ j = 1, 2,\ \ \partial_s u^+ + J(\partial_t u^+ - X_{\kappa_R^+ H_3}(u^+)) = 0,\\ &\partial_s u + J\partial_t u = 0 \text{ on } \Sigma_0,\ \ E(u) < +\infty,\\ &u(s, -1) \in o_M,\ u(s, 1) \in \nu^* N,\ s \in \mathbb{R},\ \ u(s, 0^-), u(s, 0^+) \in o_M,\ s \leqslant 0,\\ &u_j^-(-\infty) = \gamma_j^-(0),\ j = 1, 2,\ \ u^+(+\infty) = \gamma^+(0),\end{aligned}$$

where $R > 2$. A function $\kappa_R^- : (-\infty, 0] \to [0, 1]$ is defined by

$$\kappa_R^-(s) = \begin{cases} 1, & -R \leqslant s \leqslant -2 \\ 0, & s \leqslant -R - 1,\ s \geqslant -1, \end{cases}$$

and $\kappa_R^+ : [0, +\infty) \to [0, 1]$, $\kappa_R^+(s) = \kappa_R^-(-s)$. We include $(\vec{f}, \vec{H})$ in the notation for manifold $\mathcal{M}_R^{\mathrm{prod}}$ in order to emphasize that we have different functions, f_j, and Hamiltonians, H_j, at appropriate ends. Another moduli space is

$$\begin{aligned}&\mathcal{M}^{\mathrm{prod}}(p_1^-, p_2^-, p^+; \vec{f}, \vec{H}) = \\ &\qquad \{(R, \gamma_1^-, \gamma_2^-, \gamma^+, u) \mid R > R_0,\ (\gamma_1^-, \gamma_2^-, \gamma^+, u) \in \mathcal{M}_R^{\mathrm{prod}}(p_1^-, p_2^-, p^+; \vec{f}, \vec{H})\}.\end{aligned}$$

The boundary of one-dimensional component of $\mathcal{M}^{\mathrm{prod}}$ is of the form

$$\partial \mathcal{M}^{\mathrm{prod}}(p_1^-, p_2^-, p^+; \vec{f}, \vec{H}) = \mathcal{M}_{R_0}^{\mathrm{prod}}(p_1^-, p_2^-, p^+; \vec{f}, \vec{H})$$

$$\cup \bigcup_{q_1^-\in \mathrm{CF}_*(f_1)} \mathcal{M}(p_1^-, q_1^-; f_1) \times \mathcal{M}^{\mathrm{prod}}(q_1^-, p_2^-, p^+; \vec{f}, \vec{H})$$
$$\cup \bigcup_{q_2^-\in \mathrm{CF}_*(f_2)} \mathcal{M}(p_2^-, q_2^-; f_2) \times \mathcal{M}^{\mathrm{prod}}(p_1^-, q_2^-, p^+; \vec{f}, \vec{H})$$
$$\cup \bigcup_{q^+\in \mathrm{CF}_*(f_3)} \mathcal{M}^{\mathrm{prod}}(p_1^-, p_2^-, q^+; \vec{f}, \vec{H}) \times \mathcal{M}(q^+, p^+; f_3)$$
$$\cup \bigcup_{x_1^-, x_2^-, x^+} \mathcal{M}(p_1^-, f_1; x_1^-, H_1) \times \mathcal{M}(p_2^-, f_2; x_2^-, H_2) \times \mathcal{M}(x_1^-, x_2^-; x^+) \times \mathcal{M}(x^+, H_3; p^+, f_3).$$

We conclude that, at the homology level, $\phi(\psi(p_1^-) \star \psi(p_2^-))$ equals homomorphism that counts the number of elements of $\mathcal{M}^{\mathrm{prod}}_{R_0}(p_1^-, p_2^-, p^+; \vec{f}, \vec{H})$. The latter is independent of the choice of Hamiltonians. We used the same idea to show that the homomorphism L, in the proof of Theorem 1.1, is independent of Hamiltonian. Therefore, we can take Hamiltonians to be zero. Homolomorphic pants with boundary on $o_M \cup \nu^*N$ are constant, thus

$$\mathcal{M}^{\mathrm{prod}}_{R_0}(p_1^-, p_2^-, p^+; \vec{f}, \vec{H} = 0) = \mathcal{M}^{\mathrm{tree}}(p_1^-, p_2^-; p^+).$$

It follows that $\alpha \cdot \beta = \Phi(\Psi(\alpha) \star \Psi(\beta))$, for $\alpha \in \mathrm{HM}_*(M)$ and $\beta \in \mathrm{HM}_*(N)$.

Now we prove the inequality between spectral invariants stated in Theorem 1.3. Since a concatenation does not have to be a smooth function, we can find a Hamiltonian H' that is regular, smooth and close enough to the concatenation $H_1 \sharp H_2$, i.e., $\|H' - H_1\sharp H_2\|_{C^0} < \varepsilon$. The first step is to prove that the product $\star$ defines a product

$$\mathrm{CF}_*^{\lambda}(H_1) \times \mathrm{CF}_*^{\mu}(H_2) \to \mathrm{CF}_*^{\lambda+\mu+\varepsilon}(H'),$$

on filtered complexes, for every $\varepsilon > 0$ that is small enough. Let us take a smooth family of Hamiltonians $K : \mathbb{R} \times [-1, 1] \times T^*M \to \mathbb{R}$ such that

$$K(s, t, \cdot) = \begin{cases} H_1(t + 1, \cdot), & s \leqslant -1, -1 \leqslant t \leqslant 0 \\ H_2(t, \cdot), & s \leqslant -1, 0 \leqslant t \leqslant 1 \\ \frac{1}{2} H'(\frac{t+1}{2}, \cdot), & s \geqslant 1. \end{cases}$$

We can choose K such that $\left\|\frac{\partial K}{\partial s}\right\| \leqslant \varepsilon$, $s \in [-1, 1]$, and $\frac{\partial K}{\partial s} = 0$, elsewhere. Let us take $x_1^- \in \mathrm{CF}_*^{\lambda}(H_1)$ and $x_2^- \in \mathrm{CF}_*^{\mu}(H_2)$. Assume that there exists an element $u \in \mathcal{M}(x_1^-, x_2^-; x^+)$ for some $x^+ \in \mathrm{CF}_*(H')$ (u is a solution of the equation $\bar\partial_{K,J}(u) = 0$). Then it holds

$$0 \leqslant \int_\Sigma \left\|\frac{\partial u}{\partial s}\right\|_J^2 ds\,dt = \int_\Sigma \omega\left(\frac{\partial u}{\partial s}, J\frac{\partial u}{\partial s}\right) ds\,dt \tag{5.1}$$
$$= \int_\Sigma \omega\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} - X_K(u)\right) ds\,dt$$
$$= \int_\Sigma u^*\omega - \int_\Sigma dK\left(\frac{\partial u}{\partial s}\right) ds\,dt.$$

Using Stoke's formula we obtain $\int_\Sigma u^*\omega = -\int x_1^{-*}\lambda - \int x_2^{-*}\lambda + \int x^{+*}\lambda$. Using the equality

$$\int_\Sigma \frac{\partial}{\partial s}(K\circ u)\,ds\,dt = \int_\Sigma dK\left(\frac{\partial u}{\partial s}\right)ds\,dt + \int_\Sigma \frac{\partial K}{\partial s}(u)\,ds\,dt,$$

and Stoke's formula again, we get the estimate

$$\begin{aligned} -\int_\Sigma dK\left(\frac{\partial u}{\partial s}\right)ds\,dt \leqslant \int_0^1 & H_1(x_1^-(t),t)\,dt \\ &+ \int_0^1 H_2(x_2^-(t),t)\,dt - \int_0^1 H'(x^+(t),t)\,dt + 4\varepsilon. \end{aligned}$$

Thus $\mathcal{A}_{H'}(x^+) \leqslant \mathcal{A}_{H_1}(x_1^-) + \mathcal{A}_{H_2}(x_2^-) + 4\varepsilon$. From the definition of the operation $\cdot$, it easily follows

$$l(\alpha\cdot\beta; o_M, \nu^*N : H') \leqslant l(\alpha; o_M, o_M : H_1) + l(\beta; o_M, \nu^*N : H_2) + 4\varepsilon.$$

We know [**23**] that spectral invariants are continuous with respect to Hamiltonians. If we pass to the limit as $\varepsilon \to 0$, then we get the triangle inequality

$$l(\alpha\cdot\beta; o_M, \nu^*N : H_1\sharp H_2) \leqslant l(\alpha; o_M, o_M : H_1) + l(\beta; o_M, \nu^*N : H_2). \qquad \square$$

Proof of Theorem 1.4. For $x\in \mathrm{CF}_*(o_M, \nu^*N : H)$ and $y \in \mathrm{CF}_*(o_M, o_M : H)$, we define an auxiliary moduli space $\mathcal{M}^j(x,y;H)$ to be the set of all the maps $u\colon \mathbb{R}\times[0,1]\to T^*M$ such that

$$\begin{gathered} \partial_s u + J(\partial_t u - X_H(u)) = 0,\;\; u(s,0)\in o_M,\;\; s\in\mathbb{R},\;\; u(s,1)\in \nu^*N,\;\; s\leqslant 0, \\ u(s,1)\in o_M,\;\; s\geqslant 0,\;\; u(-\infty,t) = x(t),\;\; u(+\infty,t) = y(t). \end{gathered}$$

Strips of this type, with a jump on the boundary, were discussed in [**1**]. On the generators of $\mathrm{CF}_*(o_M, \nu^*N : H)$ we define m to be $m(x) = \sum_y \sharp_2\mathcal{M}^j(x,y;H)\,y$. The boundary of the one-dimensional component of $\mathcal{M}^j(x,y;H)$ is

$$\begin{aligned} \partial\mathcal{M}^j_{[1]}(x,y;H) = & \bigcup_{x'\in \mathrm{CF}_*(o_M,\nu^*N:H)} \mathcal{M}(x,x';H)\times\mathcal{M}^j(x',x;H) \\ & \cup \bigcup_{y'\in\mathrm{CF}_*(o_M,o_M:H)} \mathcal{M}^j(x,y';H)\times\mathcal{M}(y',y;H). \end{aligned}$$

Thus, m induces a map on homology level (denoted by m, again)

$$m : \mathrm{HF}_*(o_M, \nu^*N : H) \to \mathrm{HF}_*(o_M, o_M : H).$$

We can explicitly describe the induced morphism on the Morse side

$$\Phi\circ m\circ\Psi : \mathrm{HM}_*(N)\to\mathrm{HM}_*(M).$$

In order to do this, we need to correlate somehow the Morse functions on N and M. Let us take a Morse function $f : N\to\mathbb{R}$. Following [**32**], we can find a Morse function $F : M\to\mathbb{R}$ extending f, $F|_N = f$, in such a way that there are no trajectories for the negative gradient flow of F leaving N (see [**32**, Proposition 4.16 and Corollary 4.17]).

On chain complexes, $\phi \circ m \circ \psi$ is

$$\phi(m(\psi(p))) = \sum_{x,y,q} \sharp_2 \mathcal{M}(p,f,g;x,H,J) \sharp_2 \mathcal{M}^j(x,y;H) \sharp_2 \mathcal{M}(y,H,J;q,F,g)\, q,$$

where the summation is taken over

$$x \in \mathrm{CF}_k(o_M, \nu^* N : H), \;\; y \in \mathrm{CF}_k(o_M, o_M : H), \;\; q \in \mathrm{CM}_k(M : F).$$

Note that g is a metric on M. We will use the same idea as in the proof of Theorem 1.1, when we defined $\mathcal{M}_R(p,q,f;H)$ and $\overline{\mathcal{M}}(p,q,f;H)$. So, we define two auxiliary manifolds. One of them, denoted by $\mathcal{M}_R^{\mathrm{aux}}(p,f;q,F;H)$, is defined as the set of triples (γ_-, u, γ_+) such that

$$\begin{gathered}
\gamma_- : (-\infty, 0] \to N, \; dot\gamma_- = -\nabla f(\gamma_-), \;\; \gamma_+ : [0,+\infty) \to M, \; \dot{\gamma}_+ = -\nabla F(\gamma_+), \\
u : \mathbb{R} \times [0,1] \to T^*M, \;\; \partial_s u + J(\partial_t u - X_{\sigma_R H}(u)) = 0, \\
u(s,0) \in o_M, \; s \in \mathbb{R}, \;\; u(s,1) \in \nu^* N, \; s \leqslant 0, \;\; u(s,1) \in o_M, \; s \geqslant 0, \\
\gamma_-(-\infty) = p, \; \gamma_+(+\infty) = q, \;\; u(-\infty) = \gamma_-(0), \; u(+\infty) = \gamma_+(0).
\end{gathered}$$

The dimension of $\mathcal{M}_R^{\mathrm{aux}}(p,f;q,F;H)$ is $m_f(p) - m_F(q)$. The other manifold is

$$\mathcal{M}^{\mathrm{aux}}(p,f;q,F;H) = \{(R,\gamma_-,u,\gamma_+) \mid (\gamma_-,u,\gamma_+) \in \mathcal{M}_R^{\mathrm{aux}}(p,f;q,F;H), \; R > R_0\},$$

of dimension $m_f(p) - m_F(q) + 1$. For $p \in \mathrm{CM}_k(N : f)$ and $q \in \mathrm{CM}_k(M : F)$, the boundary of the one-dimensional manifold $\mathcal{M}^{\mathrm{aux}}(p,f;q,F;H)$ is

$$\begin{aligned}
\partial \mathcal{M}^{\mathrm{aux}}(p,f;q,F;H) &= \mathcal{M}_{R_0}^{\mathrm{aux}}(p,f;q,F;H) \\
&\cup \bigcup_{r \in \mathrm{CM}_{k-1}(N:f)} \mathcal{M}(p,r;f) \times \mathcal{M}^{\mathrm{aux}}(r,f;q,F;H) \\
&\cup \bigcup_{s \in \mathrm{CM}_{k+1}(M:F)} \mathcal{M}^{\mathrm{aux}}(p,f;s,F;H) \times \mathcal{M}(s,q;F) \\
&\cup \bigcup_{x \in \mathrm{CF}_k, y \in \mathrm{CF}_k} \mathcal{M}(p,f,g;x,H,J) \times \mathcal{M}^j(x,y;H) \times \mathcal{M}(y,H,J;q,F,g).
\end{aligned}$$

Thus, $\phi \circ m \circ \psi$ is chain homotopic to the map $\eta : \mathrm{CM}_k(N : f) \to \mathrm{CM}_k(M : F)$, defined by

$$\eta(p) = \sum_q \sharp_2 \mathcal{M}_{R_0}^{\mathrm{aux}}(p,f;q,F;H)\, q.$$

This map is an analogue of the map l defined in the proof of Theorem 1.1. In the same way, η is going to be chain homotopic to the map η_0 that counts combined object (γ_-, u, γ_+) where u is a holomorphic disc (perturbed by zero Hamiltonian) with the boundary on $o_M \cup \nu^* N$. We have already showed that all such discs are constant, thus η_0 counts the number of gradient trajectories of F (since $F = f$ on N) that connect $p \in N$ with some $q \in \mathrm{CM}_k(M : F)$. We assume that there are no negative gradient trajectories of F leaving N. Since p and q are of the same Morse index, a gradient trajectory connecting p and q does not exist when $p \neq q$. We conclude that $\eta_0 = i : \mathrm{CM}_k(N : f) \to \mathrm{CM}_k(M : F)$, is the inclusion of chain complexes. Once again, we construct F as an extension of f. Thus the inclusion

i of chain complexes makes sense in this situation. In Morse homology, $\Phi \circ m \circ \Psi$ and i induce the same map.

We are only left to prove inequality (1.6) among spectral invariants. Using the same idea as in (5.1), one can prove that the action functional $\mathcal{A}_H$ decreases along the holomorphic strip $u \in \mathcal{M}^j(x, y; H)$. It means that m induces the homomorphism

$$m : \mathrm{HF}_*^\lambda(o_M, \nu^*N : H) \to \mathrm{HF}_*^\lambda(o_M, o_M : H).$$

on filtered homology. So, if $\Psi(\alpha)$ is realized as an element from $\mathrm{HF}_*^\lambda(o_M, \nu^*N : H)$, then an element $m(\Psi(\alpha)) = \Psi(\Phi(m(\Psi(\alpha))))$, is realized as an element from $\mathrm{HF}_*^\lambda(o_M, o_M : H)$. The inequality (1.6) directly follows. $\square$

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