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COHOMOLOGICAL DIMENSIONS WITH RESPECT TO SUM AND INTERSECTION OF IDEALS

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Abstract. Let R be a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ and $\mathfrak{b}$ proper ideals of R, M a finitely generated R-module with finite projective dimension, and X a finitely generated R-module. We study the cohomological dimensions of M and X with respect to $\mathfrak{a}+\mathfrak{b}$ and $\mathfrak{a}\cap\mathfrak{b}$. We show that the inequality $\mathrm{cd}_{\mathfrak{a}+\mathfrak{b}}(M,X) \leqslant \mathrm{cd}_{\mathfrak{a}}(M,X) + \mathrm{cd}_{\mathfrak{b}}(X)$ holds true and we find an equivalent condition for it to be equality.

1. Introduction

Throughout R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ and $\mathfrak{b}$ are proper ideals of R, M is a finite R-module (i.e., finitely generated) and X is an arbitrary R-module which is not necessarily finite. For the basic results, notations and terminology not given in this paper, the reader is referred to [**3, 4**].

The ith generalized local cohomology module

$$\mathrm{H}^i_{\mathfrak{a}}(M,X) \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(M/\mathfrak{a}^nM, X),$$

which is a generalization of the ith ordinary local cohomology module

$$\mathrm{H}^i_{\mathfrak{a}}(X) \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, X),$$

was introduced by Herzog in his Habilitation [**7**] and then continued by Suzuki [**8**], Bijan-Zadeh [**2**], Yassemi [**9**] and some other authors. In the study of vanishing of generalized local cohomology modules, Amjadi and Naghipour in [**1**] extended the cohomological dimension of X with respect to $\mathfrak{a}$,

$$\mathrm{cd}_{\mathfrak{a}}(X) := \sup\{i : \mathrm{H}^i_{\mathfrak{a}}(X) \neq 0\},$$

to the cohomological dimension of M and X with respect to $\mathfrak{a}$,

$$\mathrm{cd}_{\mathfrak{a}}(M,X) := \sup\{i : \mathrm{H}^i_{\mathfrak{a}}(M,X) \neq 0\},$$

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and proved some basic results of this notion.

Our main ideas come from [**6**] where, for a finite R-module X, Dibaei and the author proved the inequalities

$$\begin{aligned}\operatorname{cd}_{\mathfrak{a}+\mathfrak{b}}(X) &\leqslant \operatorname{cd}_{\mathfrak{a}}(X)+\operatorname{cd}_{\mathfrak{b}}(X),\\ \operatorname{cd}_{\mathfrak{a}\cap\mathfrak{b}}(X) &\leqslant \operatorname{cd}_{\mathfrak{a}}(X)+\operatorname{cd}_{\mathfrak{b}}(X),\end{aligned} \tag{1.1}$$

and found some equivalent conditions for (1.1) to be equality [**6**, Corollaries 2.2 and 2.6] (see also [**5**, Corollary 4.2]). Here, we first present some technical results (Lemmas 2.1, 2.2, and 2.3) to generalize the above inequalities. More precisely, in Theorem 2.1 and Corollary 2.1, we show that, for a finite R-module M with finite projective dimension and for a finite R-module X, the inequality

$$\operatorname{cd}_{\mathfrak{a}+\mathfrak{b}}(M,X) \leqslant \operatorname{cd}_{\mathfrak{a}}(M,X)+\operatorname{cd}_{\mathfrak{b}}(X), \tag{1.2}$$

and one of the inequalities

$$\begin{aligned}\operatorname{cd}_{\mathfrak{a}\cap\mathfrak{b}}(M,X) &\leqslant \operatorname{cd}_{\mathfrak{a}}(M,X)+\operatorname{cd}_{\mathfrak{b}}(X),\\ \operatorname{cd}_{\mathfrak{a}\cap\mathfrak{b}}(M,X) &\leqslant \operatorname{cd}_{\mathfrak{a}}(X)+\operatorname{cd}_{\mathfrak{b}}(M,X),\end{aligned}$$

holds true. Then, in Theorem 2.2, we show that there exists an isomorphism between the R-modules $\mathrm{H}_{\mathfrak{a}+\mathfrak{b}}^{\operatorname{cd}_{\mathfrak{a}}(M,X)+\operatorname{cd}_{\mathfrak{b}}(X)}(M,X)$ and $\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}_{\mathfrak{a}}(M,X)}(M,\mathrm{H}_{\mathfrak{b}}^{\operatorname{cd}_{\mathfrak{b}}(X)}(X))$. Finally, in Corollary 2.2, we find an equivalent condition for (1.2) to be equality.

Even though we can show some of our results by using spectral sequences, we are avoiding the use of this technique completely in this work and we provide more elementary proofs for the results.

2. Main results

Our method to prove some results of the paper is based on the induction argument and we need the following useful lemma for the base cases and the inductive steps. Note that, for all i, we have the isomorphism

$$\mathrm{H}_{\mathfrak{a}}^{i}(M,X) \cong \mathrm{H}^{i}(\Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(M,E^{\bullet}))),$$

where $E^{\bullet}$ is an injective resolution of X.

Lemma 2.1. *Let M be a finite R-module, X an arbitrary R-module, and Y a $\mathfrak{b}$-torsion R-module. Then the following statements hold true.*

(i) $\Gamma_{\mathfrak{a}}(M,X) \cong \operatorname{Hom}_R(M,\Gamma_{\mathfrak{a}}(X))$.
(ii) $\Gamma_{\mathfrak{a}+\mathfrak{b}}(M,X) \cong \Gamma_{\mathfrak{a}}(M,\Gamma_{\mathfrak{b}}(X))$.
(iii) $\mathrm{H}_{\mathfrak{a}+\mathfrak{b}}^{i}(M,Y) \cong \mathrm{H}_{\mathfrak{a}}^{i}(M,Y)$ *for all* $i \geqslant 0$.

Proof. This is easy and left to the reader. □

In the next lemma, for a non-negative integer t, we find out when the generalized local cohomology module $\mathrm{H}_{\mathfrak{a}+\mathfrak{b}}^{t}(M,X)$ vanishes.

Lemma 2.2. *Let M be a finite R-module, X an arbitrary R-module, and t a non-negative integer such that $\mathrm{H}_{\mathfrak{a}}^{t-i}(M,\mathrm{H}_{\mathfrak{b}}^{i}(X))$ is zero for all i, $0 \leqslant i \leqslant t$. Then $\mathrm{H}_{\mathfrak{a}+\mathfrak{b}}^{t}(M,X)$ is also zero.*

Proof. We prove by using induction on t. The case $t = 0$ is clear from Lemma 2.1 (ii). Suppose that $t > 0$ and that $t - 1$ is settled. Let $\bar{X} = X/\Gamma_{\mathfrak{b}}(X)$ and $L = \mathrm{E}(\bar{X})/\bar{X}$ where $\mathrm{E}(\bar{X})$ is an injective hull of $\bar{X}$. Since $\Gamma_{\mathfrak{b}}(\bar{X}) = 0 = \Gamma_{\mathfrak{a}+\mathfrak{b}}(\bar{X})$, we have $\Gamma_{\mathfrak{b}}(\mathrm{E}(\bar{X})) = 0 = \Gamma_{\mathfrak{a}+\mathfrak{b}}(\mathrm{E}(\bar{X}))$ and so, by Lemma 2.1 (i), $\Gamma_{\mathfrak{b}}(\mathrm{E}(\bar{X})) = 0 = \Gamma_{\mathfrak{a}+\mathfrak{b}}(M, \mathrm{E}(\bar{X}))$. Applying the derived functors of $\Gamma_{\mathfrak{b}}(-)$ and $\Gamma_{\mathfrak{a}+\mathfrak{b}}(M, -)$ to the short exact sequence

$$0 \longrightarrow \bar{X} \longrightarrow \mathrm{E}(\bar{X}) \longrightarrow L \longrightarrow 0,$$

we obtain the isomorphisms

$$\mathrm{H}^i_{\mathfrak{b}}(L) \cong \mathrm{H}^{i+1}_{\mathfrak{b}}(\bar{X}) \cong \mathrm{H}^{i+1}_{\mathfrak{b}}(X), \tag{2.1}$$

$$\mathrm{H}^i_{\mathfrak{a}+\mathfrak{b}}(M, L) \cong \mathrm{H}^{i+1}_{\mathfrak{a}+\mathfrak{b}}(M, \bar{X}) \tag{2.2}$$

for all $i \geqslant 0$. From the isomorphisms (2.1), for all i, $0 \leqslant i \leqslant t-1$, we have

$$\mathrm{H}^{(t-1)-i}_{\mathfrak{a}}(M, \mathrm{H}^i_{\mathfrak{b}}(L)) \cong \mathrm{H}^{t-(i+1)}_{\mathfrak{a}}(M, \mathrm{H}^{i+1}_{\mathfrak{b}}(X))$$

which is zero by the assumptions. Thus, from the induction hypothesis on L, we have $\mathrm{H}^{t-1}_{\mathfrak{a}+\mathfrak{b}}(M, L) = 0$. Therefore $\mathrm{H}^t_{\mathfrak{a}+\mathfrak{b}}(M, \bar{X}) = 0$ by the isomorphisms (2.2). Now, by the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{b}}(X) \longrightarrow X \longrightarrow \bar{X} \longrightarrow 0,$$

we get the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^t_{\mathfrak{a}+\mathfrak{b}}(M, \Gamma_{\mathfrak{b}}(X)) \longrightarrow \mathrm{H}^t_{\mathfrak{a}+\mathfrak{b}}(M, X) \longrightarrow \mathrm{H}^t_{\mathfrak{a}+\mathfrak{b}}(M, \bar{X}) \longrightarrow \cdots.$$

The above long exact sequence in conjunction with Lemma 2.1 (iii) and the assumptions, show that $\mathrm{H}^t_{\mathfrak{a}+\mathfrak{b}}(M, X) = 0$. □

The following lemma is a generalization of [**1**, Theorem B] and is crucial in the rest of the paper.

Lemma 2.3. *Suppose that M is a finite R-module with finite projective dimension, X is a finite R-module, and Y is an arbitrary R-module such that* $\mathrm{Supp}_R(Y) \subseteq \mathrm{Supp}_R(X)$. *Then*

$$\mathrm{cd}_{\mathfrak{a}}(M, Y) \leqslant \mathrm{cd}_{\mathfrak{a}}(M, X).$$

Proof. Since generalized local cohomology commutes with direct limit and Y is the direct limit of its finite submodules, the assertion follows from [**1**, Theorem B]. □

The following theorem is one of our main results which generalizes [**6**, Corollary 2.2 (i)] (see also [**5**, Corollary 4.2]).

Theorem 2.1. *Let M be a finite R-module with finite projective dimension and X a finite R-module. Then we have*

$$\mathrm{cd}_{\mathfrak{a}+\mathfrak{b}}(M, X) \leqslant \mathrm{cd}_{\mathfrak{a}}(M, X) + \mathrm{cd}_{\mathfrak{b}}(X).$$

PROOF. Let t be an integer bigger than $\mathrm{cd}_{\mathfrak{a}}(M,X)+\mathrm{cd}_{\mathfrak{b}}(X)$. We prove that $\mathrm{H}^t_{\mathfrak{a}+\mathfrak{b}}(M,X)=0$. From Lemma 2.2, it is enough to show that $\mathrm{H}^{t-i}_{\mathfrak{a}}(M,\mathrm{H}^i_{\mathfrak{b}}(X))=0$ for all i, $0\leqslant i\leqslant t$. By the definition of cohomological dimension of X with respect to $\mathfrak{b}$, $\mathrm{H}^{t-i}_{\mathfrak{a}}(M,\mathrm{H}^i_{\mathfrak{b}}(X))=0$ if $i>\mathrm{cd}_{\mathfrak{b}}(X)$. Otherwise, $t-i>\mathrm{cd}_{\mathfrak{a}}(M,X)$. Since $\mathrm{Supp}_R(X)\supseteq\mathrm{Supp}_R(\mathrm{H}^i_{\mathfrak{b}}(X))$, $\mathrm{cd}_{\mathfrak{a}}(M,X)\geqslant\mathrm{cd}_{\mathfrak{a}}(M,\mathrm{H}^i_{\mathfrak{b}}(X))$ from Lemma 2.3. Therefore $\mathrm{H}^{t-i}_{\mathfrak{a}}(M,\mathrm{H}^i_{\mathfrak{b}}(X))=0$ by the definition of cohomological dimension of M and $\mathrm{H}^i_{\mathfrak{b}}(X)$ with respect to $\mathfrak{a}$. □

The following corollary is an immediate application of the above theorem which generalizes [**6**, Corollary 2.2 (ii)].

COROLLARY 2.1. *Let M be a finite R-module with finite projective dimension and X a finite R-module. Then one of the following inequalities holds true.*

(i) $\mathrm{cd}_{\mathfrak{a}\cap\mathfrak{b}}(M,X)\leqslant\mathrm{cd}_{\mathfrak{a}}(M,X)+\mathrm{cd}_{\mathfrak{b}}(X)$ *or*
(ii) $\mathrm{cd}_{\mathfrak{a}\cap\mathfrak{b}}(M,X)\leqslant\mathrm{cd}_{\mathfrak{a}}(X)+\mathrm{cd}_{\mathfrak{b}}(M,X)$.

PROOF. By considering the Mayer–Vietoris exact sequence

$$\begin{aligned}0\longrightarrow\Gamma_{\mathfrak{a}+\mathfrak{b}}(M,X)&\longrightarrow\Gamma_{\mathfrak{a}}(M,X)\oplus\Gamma_{\mathfrak{b}}(M,X)\\&\longrightarrow\Gamma_{\mathfrak{a}\cap\mathfrak{b}}(M,X)\longrightarrow\mathrm{H}^1_{\mathfrak{a}+\mathfrak{b}}(M,X)\longrightarrow\cdots\\&\longrightarrow\mathrm{H}^i_{\mathfrak{a}+\mathfrak{b}}(M,X)\longrightarrow\mathrm{H}^i_{\mathfrak{a}}(M,X)\oplus\mathrm{H}^i_{\mathfrak{b}}(M,X)\\&\longrightarrow\mathrm{H}^i_{\mathfrak{a}\cap\mathfrak{b}}(M,X)\longrightarrow\mathrm{H}^{i+1}_{\mathfrak{a}+\mathfrak{b}}(M,X)\longrightarrow\cdots\end{aligned}$$

and Theorem 2.1, we get the inequality $\mathrm{cd}_{\mathfrak{a}\cap\mathfrak{b}}(M,X)\leqslant\mathrm{cd}_{\mathfrak{a}}(M,X)+\mathrm{cd}_{\mathfrak{b}}(X)$ if $\mathrm{cd}_{\mathfrak{a}}(M,X)\geqslant\mathrm{cd}_{\mathfrak{b}}(M,X)$, and the inequality $\mathrm{cd}_{\mathfrak{a}\cap\mathfrak{b}}(M,X)\leqslant\mathrm{cd}_{\mathfrak{a}}(X)+\mathrm{cd}_{\mathfrak{b}}(M,X)$ if $\mathrm{cd}_{\mathfrak{a}}(M,X)\leqslant\mathrm{cd}_{\mathfrak{b}}(M,X)$. □

In the following lemma, for non-negative integers s and t, we find some sufficient conditions for validity of the isomorphism $\mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(M,X)\cong\mathrm{H}^s_{\mathfrak{a}}(M,\mathrm{H}^t_{\mathfrak{b}}(X))$.

LEMMA 2.4. *Let M be a finite R-module, X an arbitrary R-module, and s,t non-negative integers such that*

(i) $\mathrm{H}^{s+t-i}_{\mathfrak{a}}(M,\mathrm{H}^i_{\mathfrak{b}}(X))=0$ *for all* $i\neq t$,
(ii) $\mathrm{H}^{s+t+1-i}_{\mathfrak{a}}(M,\mathrm{H}^i_{\mathfrak{b}}(X))=0$ *for all* $i<t$,
(iii) $\mathrm{H}^{s+t-1-i}_{\mathfrak{a}}(M,\mathrm{H}^i_{\mathfrak{b}}(X))=0$ *for all* $i>t$.

Then we have

$$\mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(M,X)\cong\mathrm{H}^s_{\mathfrak{a}}(M,\mathrm{H}^t_{\mathfrak{b}}(X)).$$

PROOF. Let $\bar{X}=X/\Gamma_{\mathfrak{b}}(X)$ and $L=\mathrm{E}(\bar{X})/\bar{X}$ where $\mathrm{E}(\bar{X})$ is an injective hull of $\bar{X}$. We prove by using induction on t. In the case that $t=0$, we have $\mathrm{H}^{s-1}_{\mathfrak{a}+\mathfrak{b}}(M,\bar{X})=0=\mathrm{H}^s_{\mathfrak{a}+\mathfrak{b}}(M,\bar{X})$ from hypothesis (iii) and (i), and Lemma 2.2. Now, the assertion follows by the exact sequence

$$\mathrm{H}^{s-1}_{\mathfrak{a}+\mathfrak{b}}(M,\bar{X})\longrightarrow\mathrm{H}^s_{\mathfrak{a}+\mathfrak{b}}(M,\Gamma_{\mathfrak{b}}(X))\longrightarrow\mathrm{H}^s_{\mathfrak{a}+\mathfrak{b}}(M,X)\longrightarrow\mathrm{H}^s_{\mathfrak{a}+\mathfrak{b}}(M,\bar{X}),$$

obtained from the short exact sequence

$$0\longrightarrow\Gamma_{\mathfrak{b}}(X)\longrightarrow X\longrightarrow\bar{X}\longrightarrow 0,$$

and Lemma 2.1 (iii).

Suppose that $t>0$ and that $t-1$ is settled. Since $\Gamma_{\mathfrak{b}}(\bar{X})=0=\Gamma_{\mathfrak{a}+\mathfrak{b}}(\bar{X})$, we have $\Gamma_{\mathfrak{b}}(\mathrm{E}(\bar{X}))=0=\Gamma_{\mathfrak{a}+\mathfrak{b}}(\mathrm{E}(\bar{X}))$ and so, by Lemma 2.1 (i), we get $\Gamma_{\mathfrak{b}}(\mathrm{E}(\bar{X}))=0=\Gamma_{\mathfrak{a}+\mathfrak{b}}(M,\mathrm{E}(\bar{X}))$. Applying the derived functors of $\Gamma_{\mathfrak{b}}(-)$ and $\Gamma_{\mathfrak{a}+\mathfrak{b}}(M,-)$ to the short exact sequence

$$0\longrightarrow \bar{X}\longrightarrow \mathrm{E}(\bar{X})\longrightarrow L\longrightarrow 0,$$

we obtain the isomorphisms

$$\mathrm{H}^{i}_{\mathfrak{b}}(L)\cong \mathrm{H}^{i+1}_{\mathfrak{b}}(\bar{X})\cong \mathrm{H}^{i+1}_{\mathfrak{b}}(X), \tag{2.3}$$

$$\mathrm{H}^{i}_{\mathfrak{a}+\mathfrak{b}}(M,L)\cong \mathrm{H}^{i+1}_{\mathfrak{a}+\mathfrak{b}}(M,\bar{X}) \tag{2.4}$$

for all $i\geqslant 0$. From isomorphisms (2.3) and the assumptions, we have

- $\mathrm{H}^{s+(t-1)-i}_{\mathfrak{a}}(M,\mathrm{H}^{i}_{\mathfrak{b}}(L))=\mathrm{H}^{s+t-(i+1)}_{\mathfrak{a}}(M,\mathrm{H}^{i+1}_{\mathfrak{b}}(X))=0$ for all $i\neq t-1$,
- $\mathrm{H}^{s+(t-1)+1-i}_{\mathfrak{a}}(M,\mathrm{H}^{i}_{\mathfrak{b}}(L))=\mathrm{H}^{s+t+1-(i+1)}_{\mathfrak{a}}(M,\mathrm{H}^{i+1}_{\mathfrak{b}}(X))=0$ for all $i<t-1$,
- $\mathrm{H}^{s+(t-1)-1-i}_{\mathfrak{a}}(M,\mathrm{H}^{i}_{\mathfrak{b}}(L))=\mathrm{H}^{s+t-1-(i+1)}_{\mathfrak{a}}(M,\mathrm{H}^{i+1}_{\mathfrak{b}}(X))=0$ for all $i>t-1$.

Thus, by the induction hypothesis on L, we get $\mathrm{H}^{s+(t-1)}_{\mathfrak{a}+\mathfrak{b}}(M,L)\cong \mathrm{H}^{s}_{\mathfrak{a}}(M,\mathrm{H}^{t-1}_{\mathfrak{b}}(L))$. Therefore $\mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(M,\bar{X})\cong \mathrm{H}^{s}_{\mathfrak{a}}(M,\mathrm{H}^{t}_{\mathfrak{b}}(X))$ from isomorphisms (2.3) and (2.4). Now, by Lemma 2.1 (iii), assumptions (i) and (ii), and the exact sequence

$$\mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(M,\Gamma_{\mathfrak{b}}(X))\longrightarrow \mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(M,X)\longrightarrow \mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(M,\bar{X})\longrightarrow \mathrm{H}^{s+t+1}_{\mathfrak{a}+\mathfrak{b}}(M,\Gamma_{\mathfrak{b}}(X)),$$

obtained from the short exact sequence

$$0\longrightarrow \Gamma_{\mathfrak{b}}(X)\longrightarrow X\longrightarrow \bar{X}\longrightarrow 0,$$

we get $\mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(M,X)\cong \mathrm{H}^{s}_{\mathfrak{a}}(M,\mathrm{H}^{t}_{\mathfrak{b}}(X))$ which completes the proof. ☐

The following theorem generalizes [**6**, Corollary 2.6 (i)]. Note that, for finite R-modules M and X, if $(\mathrm{Ann}_R(M)+\mathfrak{a}+\mathfrak{b})X\neq X$, then $(\mathrm{Ann}_R(M)+\mathfrak{a})X\neq X$ and $\mathfrak{b}X\neq X$. Thus we get $\mathrm{cd}_{\mathfrak{a}}(M,X)\geqslant 0$ and $\mathrm{cd}_{\mathfrak{b}}(X)\geqslant 0$.

Theorem 2.2. *Let M be a finite R-module with finite projective dimension and X a finite R-module with* $(\mathrm{Ann}_R(M)+\mathfrak{a}+\mathfrak{b})X\neq X$. *Then we have*

$$\mathrm{H}^{\mathrm{cd}_{\mathfrak{a}}(M,X)+\mathrm{cd}_{\mathfrak{b}}(X)}_{\mathfrak{a}+\mathfrak{b}}(M,X)\cong \mathrm{H}^{\mathrm{cd}_{\mathfrak{a}}(M,X)}_{\mathfrak{a}}(M,\mathrm{H}^{\mathrm{cd}_{\mathfrak{b}}(X)}_{\mathfrak{b}}(X)).$$

Proof. Consider Lemma 2.3 and apply Lemma 2.4 with $s=\mathrm{cd}_{\mathfrak{a}}(M,X)$ and $t=\mathrm{cd}_{\mathfrak{b}}(X)$. ☐

In the following corollary, we find an equivalent condition for

$$\mathrm{cd}_{\mathfrak{a}+\mathfrak{b}}(M,X)\leqslant \mathrm{cd}_{\mathfrak{a}}(M,X)+\mathrm{cd}_{\mathfrak{b}}(X)$$

to be equality. This corollary generalizes [**6**, Corollary 2.6 (ii)].

Corollary 2.2. *Let M be a finite R-module with finite projective dimension and X a finite R-module with* $(\mathrm{Ann}_R(M)+\mathfrak{a}+\mathfrak{b})X\neq X$. *Then the following statements are equivalent.*

(i) $\mathrm{cd}_{\mathfrak{a}+\mathfrak{b}}(M,X)=\mathrm{cd}_{\mathfrak{a}}(M,X)+\mathrm{cd}_{\mathfrak{b}}(X)$.
(ii) $\mathrm{cd}_{\mathfrak{a}}(M,X)=\mathrm{cd}_{\mathfrak{a}}(M,\mathrm{H}^{\mathrm{cd}_{\mathfrak{b}}(X)}_{\mathfrak{b}}(X))$.

Proof. (i)$\Rightarrow$ (ii). This follows from Theorem 2.2 and Lemma 2.3.
(ii)$\Rightarrow$ (i). It follows by Theorems 2.1 and 2.2. □

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