Turbulence, chaoticity and ω-limit sets

F. Balibrea and C. La Paz

Abstract

In [6] the turbulent maps have been charaterized via the topological structure of some ω-limit sets produced by them. The aim of this paper is to give a direct and easier **proof** of such a characterization and also complete it.

1 Introduction

Let $I = [0,1]$ and denote by $C(I)$ the set of continuous maps from I into I. A map $f \in C(I)$ is called turbulent if there are closed intervals J and K, with at most one point in common, such that $f(J) \cap f(K) \supseteq J \cup K$ [1].

In [2] is proved the more operative characterizacion of such a maps:

"A map $f \in C(I)$ is turbulent if and only if, there are points a, c, b $\in I$ such that:

$$a < c < b \;\; ; \;\; f(a) = f(b) = a \;\; ; \;\; f(c) = b$$

and

$$f(x) > a \text{ for } x \in (a,b) \;\; ; \;\; x < f(x) < b \text{ for } x \in (a,c)$$

or the reverse same inequalities".

The turbulent maps have a complicated dynamics in what concerns periodic structure. They have periodic points of period 3 and according to Sharkovskii's Theorem[3] have periodic points of all periods.

2 Notation and terminology

Given $x_0 \in I$ the sequence $Orb_f(x_0) = (f^n(x_0))_{n=0}^{\infty}$ where $f^0(x_0) = x_0$ and $f^n(x_0) = f \circ f^{n-1}(x_0)$ for any n, is called the orbit by f of the point x_0 and the set composed by the limit-points of $Orb_f(x_0)$ is called the ω-limit set of x_0 by f. It is used to indicate it by $\omega(x_0, f)$ and it is a non-empty closed set and invariant by f. On the other hand if $\omega(x_0, f) = A \cup B$, with A, B closed and disjoint sets then $f(A) \cap B \neq \emptyset$.

If $f^{k-1}(x_0) = x_0$ and $f^i(x_0) \neq f^{i+1}(x_0)$, $i = 0 \ldots\ldots k-2$ we will say that x_0 is a cyclic point of order k with respect to f and the set:

$$\left\{x_0, f(x_0) \ldots\ldots f^{k-1}(x_0)\right\}$$

a k-cycle.

If $f(x_0) = x_0$ that is, if x_0 is a cyclic point of order 1, we will say that x_0 is a fixed point.

With A' we denote the derivative set of A.

Given a sequence $(a_n) \subset I$ of distinct points that are not limit-points we recurrently define a transfinite sequence of sets in the following way:

$$A_0 = (a_n)$$

$$A_\alpha = \bigcap_{\beta<\alpha} A_\beta \text{ if } \alpha \text{ is a transfinite limit number}$$

$$A_\alpha = A'_{\alpha-1} \text{ in any other case}$$

It is easy to prove that for any sequence (a_n) there exists a transfinite number α such that $A_\alpha = \{p_1, p_2, ...p_n\}$.

If z is a limit-point of the set A and $(x, z) \cap A = \emptyset$ (resp. $(z, x) \cap A = \emptyset$) , then we say that z is one sided isolated limit-point from the left (resp. right).

3 Results

Theorem 3.1 *The map $f \in C(I)$ is turbulent if and only if, there exists a point $x_0 \in I$ such that $\omega(x_0, f)$ is an infinite countable set with a fixed one sided isolated limit-point.*

Proof. If part). Let suppose $a \in \omega(x_0, f)$ a fixed on sided isolated point from the left. Since f is a continuous map and a is not an attracting fixed point, then there exists $z \in \omega(x_0, f) \cap [a, 1]$ such that $f(z) > z$.

If

$$\bar{x} = \max\{x \in [a, z] \quad , \quad f(x) \leq x\} , \quad b = \min\{x \geq z \quad , \quad f(x) = \bar{x}\}$$

then there exists $c \in (\bar{x}, b)$ such that $f(c) \geq b$. On the contrary $\omega(x_0, f) \cap (\bar{x}, b) \neq \emptyset$ and $f(\bar{x}, b) \subseteq (\bar{x}, b)$ which is not possible.

Only if part)If f is a turbulent map with $a < c < b$, then the sequence (x_n) recurrently defined by:

$$x_1 = b \quad , \quad x_2 = c \quad , \quad x_{n+1} = \inf\{x \in (a, x_n) \quad , \quad f(x) = x_n\}$$

is increasing. Moreover $f(x_n) = x_{n-1}$ for all $n \geq 2$, $f(x_1) = a$.

Let be $(V_n^j)_{n,j=1}^\infty$ a basis of neighborhoods of (x_n) whose diameters are respectively less than $\frac{1}{j}$ and B_1, $B_2, \cdots$ an order of them. Given B_i , B_{i+1} there is no difficult to prove the existence of compact intervals $J_i \subset K_i \subset B_i$ and $m_i \in \mathbf{N}$ such that:

$$f^{m_i}(J_i) \subset B_{i+1} \text{ and } (f^{m_i}(J_i)) \cap (x_n) \neq \emptyset$$

If (J_n), (K_n) are the sequences of compact intervals recurrently defined in such a way that:

$$K_{n+1} = f^{m_n}(J_n) \ ; \ J_{n+1} \subset K_{n+1} \subset B_{n+1}$$

then the sequence of compact intervals $\left(f^{-\sum\limits_{i=1}^{n} m_i} (J_{i+1}) \right)$ is decreasing.

If $x_0 \in \bigcap\limits_n \left(f^{-\sum\limits_{i=1}^{n} m_i} (J_{i+1}) \right)$ then $\omega(x_0, f) = a \cup (x_n)$, the point a is fixed and $a < x_n < f(x_n)$ for $n \geq 2$.

Lemma 3.2 *If $\omega(x_0, f)$ has two fixed points then it is a turbulent map.*

Proof. If $\bar{x}$, $\bar{y}$ are two fixed points of $\omega(x_0, f)$ none of them one sided isolated, then there exists $z \in \left[\bar{x}, \bar{y}\right]$ such that $f(z) > z$ or $f(z) < z$. A similar argument to previous **theorem** ends the **proof**.

Theorem 3.3 *If $f \in C(I)$, then f^2 is turbulent if and only if there exists a point $x_0 \in I$ such that $\omega(x_0, f)$ is an infinite countable set holding at least one of the conditions:*

i)It posseses a one sided isolated cyclic point of order at most two.

ii)It posseses a fixed point.

Proof. Only if part) It is a consequence of **theorem 3.1**.

If part) i)Since $\omega(x_0, f) = \omega(x_0, f^2) \cup \omega(f(x_0), f^2)$, then the ciclyc point is one-sided isolated fixed limit-point of any of $\omega(x_0, f^2)$ or $\omega(f(x_0), f^2)$ and then f^2 is a turbulent map.

ii)If there exist two fixed points of f in $\omega(x_0, f)$ or a unique fixed point $\bar{x}$ and $z_1 \in [0, \bar{x}]$ or $z_2 \in [\bar{x}, 1]$ such that $f(z_1) < z_1$ or $f(z_2) > z_2$, then f and f^2 are turbulent maps.

If there exists $\bar{x}$, a unique not one sided isolated fixed point of $\omega(x_0, f)$ and $f^2\left[0, \bar{x}\right] \cap \omega(x_0, f) \subseteq \left[0, \bar{x}\right]$, $f^2\left[\bar{x}, 1\right] \cap \omega(x_0, f) \subseteq \left[\bar{x}, 1\right]$, then there exists $z \in \left[\bar{x}, 1\right]$ such that $f^2(z) = \max \omega(x_0, f)$ and using a similar argument to that **theorem 3.1** the **proof** is over.

Theorem 3.4 *The topological entropy* [4]*of $f \in C(I)$ $(h(f))$ is positive if and only if there exists $x_0 \in I$ such that $\omega(x_0, f)$ is a countable infinite set.*

Proof. If $h(f) > 0$ then f^n is a turbulent map for some n [5]. Therefore there exists $x_0 \in I$ such that $\omega(x_0, f^n)$ is a infinte countable set. Consequently $\omega(x_0, f)$ is also a infinite countable set.

If $\omega(x_0, f)$ is a infinite countable set then $(\omega(x_0, f))_\alpha$ is a finite set formed by a union of cycles of f, Thus there exists $n \in \mathbf{N}$ such that f^n have at least a fixed point. According to **theorem 3.3** the map f^{2n} is turbulent and $h(f) > 0$.

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Departamento de Matemáticas.
Departamento de Matemática Aplicada y Estadística.
Universidad de Murcia.