Existence and Uniqueness Results for a Coupled Problem Related to the Stationary Navier-Stokes System

B. CLIMENT and E. FERNÁNDEZ-CARA *

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Abstract

In this paper, we consider some systems which are close to the stationary Navier-Stokes equations. The structure of these systems is the following: An N-dimensional equation for motion, the incompressibility condition and a scalar equation involving an additional unknown, $k = k(x)$. Among other things, they serve to model the behavior of certain turbulent flows. Our main interest concerns existence and uniqueness. The main difficulties are due to the structure of the scalar equation; in particular, the right side is typically in L^1 and, furthermore, there are nonlinear terms of the kind $\nabla \cdot (\mu(k)\nabla k)$ and $\nabla \cdot (B(k))$, where μ and B are general continuous functions.

*Department of Differential Equations and Numerical Analysis, University of Sevilla, Tarfia s/n, E-41012 Sevilla, Spain. Partially supported by D.G.I.C.Y.T. (Spain), Proyecto PB92–0696.

Notation:

- $L^1 = L^1(\Omega)$, $\quad H_0^1 = H_0^1(\Omega)$, etc.
- $|\cdot|$ (resp. $\|\cdot\|$) denotes the usual norm in L^2 (resp. H_0^1).
- $H^{-1} = H^{-1}(\Omega)$ is the dual space of H_0^1 ; $\|\cdot\|_*$ denotes the usual norm in H^{-1}.
- $z_+ = \max(z, 0)$ for any real z .
- $T_M(s) = s$ if $s \in [-M, M]$; $T_M(s) = M \operatorname{sign} s$ otherwise.
- L_n is the piecewise linear even function satisfying $L_n(s) = 1$ if $s \in [0, n]$, $L_n(s) = \frac{s}{n} + 2$ if $s \in [n, 2n]$ and $L_n(s) = 0$ if $s > 2n$.
- $S : D = \sum_{i,j=1}^{N} S_{ij} D_{ij}$ for any $S = \{S_{ij}\}$ and $D = \{D_{ij}\}$.
- N' is the conjugate exponent of N , i.e. $N' = \frac{N}{N-1}$.
- For each $p \in [1, \infty]$, p^* is the associated Sobolev embedding exponent: $p^* = \frac{Np}{N-p}$ if $p < N$; $1 < p^* < \infty$ is arbitrary if $p = N$ and $p^* = \infty$ otherwise. In particular, $(N')^* = \frac{N}{N-2}$ if $N \geq 3$.

1 Introduction. Description of the problem

This paper is concerned with some nonlinear partial differential systems stemming from fluid mechanics. These are variants of the stationary Navier-Stokes equations and read as follows:

$$\left\{\begin{array}{l} -\nabla\cdot(\nu Du + k\Phi'(Du)) + (u\cdot\nabla)u + \nabla p = f\,, \\ \ \nabla\cdot u = 0\,, \\ -\nabla\cdot(\mu(k)\nabla k + B(k)) + u\cdot\nabla k = \nu'|Du|^2 + k\Phi'(Du) : Du \\ \qquad\qquad\qquad\qquad\qquad\qquad - |k|^{1/2}k\psi^0(Du)\,. \end{array}\right. \tag{1}$$

In (1), it is assumed that $Du = \nabla u + {}^t\nabla u$. The functions $D \mapsto \Phi(D)$, $D \mapsto \psi^0(D)$, $k \mapsto \mu(k)$ and $k \mapsto B(k)$ are given. Once an open set $\Omega \subset \mathbb{R}^N$ and the data $\nu > 0$, $\nu' \in [0,\nu]$ and f are fixed, we search for a solution $\{u,p,k\}$ to (1), together with appropriate boundary value conditions.

Systems like (1) are motivated by turbulence modelling. More precisely, let $U = U(x,t)$ and $P = P(x,t)$ be respectively the velocity field and pressure distribution of a viscous incompressible fluid in turbulent regime. Then, the couple (U,P) must satisfy the instationary Navier-Stokes equations. Denoting by u and p the corresponding time-averaged variables (that is to say, $u = \overline{U}$ and $p = \overline{P}$) and setting

$$U = u + u', \qquad P = p + p',$$

it is customary to replace the search of a solution to (1) by the analysis of a system that should be satisfied by u and p. After some computations, one finds:

$$-\nabla\cdot(\nu Du + R) + (u\cdot\nabla)u + \nabla p = f\,, \qquad \nabla\cdot u = 0\,, \tag{2}$$

where f is the time-averaged external forces field acting on the fluid particles and R is the so called Reynolds tensor:

$$R = \{R_{ij}\}\,, \quad \text{with} \quad R_{ij} = -\overline{u_i'u_j'}\,.$$

Since in (2) we still find the unknown variables u_i', it is reasonable to introduce closing hypotheses relating R to u. In the case of usual one-equation models, one imposes the following hypothesis of the Boussinesq kind:

$$R = \nu_T\, Du\,, \qquad \text{where} \quad \nu_T = F(k) \qquad \text{(an algebraic relation)}. \tag{3}$$

Here, $k = \overline{\frac{1}{2}|u'|^2}$ is the mean turbulent kinetic energy. The problem is thus closed using (2), (3) and an additional PDE for k.

Unfortunately, when one tries to deduce an equation for k, one finds again terms in which the turbulent perturbations u_i' (and k') appear. More precisely, one has:

$$- \nabla \cdot \left(\nu \nabla k + (\overline{-(p' + k')u'}) \right) + u \cdot \nabla k = R : Du - \overline{\tfrac{\nu}{2} |Du'|^2} . \qquad (4)$$

Consequently, one has to replace (4) by an approximation. This is made by introducing new closing hypotheses:

- There is general agreement in the approximation of the dissipation term $\overline{\frac{\nu}{2}|Du'|^2}$. It is usually replaced by a constant times $k^{3/2}$.
- Of course, (3) is used again in order to approximate the production term $R : Du$.
- The approximation of $\overline{-(p' + k')u'}$ has been achieved by several authors in different ways. In most papers, this term is replaced by $c\nu_T \nabla k$, where c is an experimental constant (for instance, see [13], [12] and the references therein). In others, it is replaced by a vector $B(k)$ (see [7]).

Hence, it is clear that equations like (1) can be used to describe the behavior of certain turbulent flows. Another motivation for (1) can be found in non Newtonian mechanics. In this setting, $\{u, p\}$ are the true velocity field and pressure, k is the temperature and it is assumed that the stress tensor τ depends on Du and k as follows:

$$\tau = \nu Du + k\Phi'(Du) . \qquad (5)$$

2 The main results

In the sequel, we will consider a simplified version of (1):

$$\begin{cases} -\nu \Delta u - \nabla \cdot (k\Phi'(\nabla u)) + (u \cdot \nabla)u + \nabla p = f \, , \\ \nabla \cdot u = 0 \, , \\ -\nabla \cdot (\mu(k)\nabla k + B(k)) + u \cdot \nabla k = \nu |\nabla u|^2 + k\Phi'(\nabla u) : \nabla u \, . \end{cases} \qquad (6)$$

This is made for convenience only; the results in this section also hold for (1) with appropriate changes. In (6), the first, second and third equations will be respectively known as the motion equation, the incompressibility condition and the energy equation. Our assumptions are the following:

- $\Omega \subset \mathbb{R}^N$ is a bounded, connected, open and regular set; $\nu > 0$ and $f \in H^{-1}$.

- $D \mapsto \Phi(D)$ is C^1, $\Phi'(0) = 0$, $|\Phi'(D)| \leq$ Const. and $D \mapsto \Phi'(D) : D$ is convex (consequently, it is also locally Lipschitz-continuous). In particular, $D \mapsto \Phi(D)$ is convex and one has $(\Phi'(D_1) - \Phi'(D_2)) : (D_1 - D_2) \geq 0$ for all D_1 and D_2.

- $k \mapsto \mu(k)$ and $k \mapsto B(k)$ are continuous functions; furthermore, $\mu(k) \geq \mu_0 > 0$ for all k.

We want to solve (6) together with Dirichlet conditions for u and k:

$$u = 0 \quad \text{and} \quad k = 0 \quad \text{on} \quad \partial\Omega\,. \tag{7}$$

Our main interest concerns general continuous functions μ and B. This is of course motivated by the fact that, in turbulence modelling, an equation exactly satisfied by k is unknown. Besides the usual spaces L^2, H_0^1, V, etc., we will use the following:

$$\mathcal{L} = \{\, \psi \in L^1\,;\ T_M(\psi) \in H_0^1 \quad \forall M > 0\,, \quad \lim_{n\to\infty} \frac{1}{n} \int_{n \leq |\psi| \leq 2n} |\nabla\psi|^2\,dx = 0\,\}$$

(see the Notation).

Theorem 1– *Under the previous assumptions, there exists $\{u, p, k\}$, with $u \in V$, $p \in L^2$ and $k \in \mathcal{L}$ such that:*

1. *The couple $\{u, p\}$ solves the first two equations in* (6) *in the usual weak or distributional sense.*

2. *$k \geq 0$ and solves the third equation in* (6) *in the following sense:*

$$\left\{ \begin{array}{l} -\nabla \cdot (\beta(k)(\mu(k)\nabla k + B(k))) + \beta'(k)\nabla k \cdot (\mu(k)\nabla k + B(k)) \\ \quad + \beta(k)(u \cdot \nabla k) = \beta(k)\,(\nu|\nabla u|^2 + k\Phi'(\nabla u) : \nabla u) \end{array} \right. \tag{8}$$

in $\mathcal{D}'(\Omega)$ for every $\beta \in W^{1,\infty}(\mathbb{R})$ with compact support.

A triplet $\{u, p, k\}$ as above will be called a weak-renormalized solution to (6). Renormalized solutions to PDE's seem to have been introduced by R. DiPerna and P.L. Lions in [8], in the framework of the Boltzmann equation. They have been used in connection with various nonlinear elliptic equations by P. Benilan et al. [3], L. Boccardo et al. [6] and P.L. Lions and F. Murat [10] (see also [11]). In the analysis of existence results for problems similar to (1) and (6), weak-renormalized solutions were considered by R. Lewandowski [9] (see also [2]). That we look for a renormalized solution

k is motivated by the structure of the right side of the energy eqution in (6) (typically in L^1) and also by our interest in keeping μ and B as general as possible.

Let us denote by $\tilde{\mu}$ the following function:

$$\tilde{\mu}(s) = \int_0^s \mu(\sigma)\, d\sigma \qquad \forall s \in \mathbb{R}\,.$$

THEOREM 2– *Assume that, in theorem 1, one has $B \equiv 0$. Then the solution $\{u, p, k\}$ furnished by theorem 1 satisfies*

$$\tilde{\mu}(k) \in \bigcap_{q<N'} W_0^{1,q}, \quad \nabla\tilde{\mu}(k) = \mu(k)\nabla k \tag{9}$$

and also the following:

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega \mu(k)\nabla k \cdot \nabla\phi + \int_\Omega (u\cdot\nabla k)\phi = \int_\Omega \Big(\nu|\nabla u|^2 + k\Phi'(\nabla u) : \nabla u\Big)\,\phi \\ \forall\phi \in \mathcal{D}(\Omega)\,. \end{array} \right. \tag{10}$$

If $\{u, p, k\}$ is as in theorem 2, it will be said it is a weak solution to (6).

THEOREM 3– *Assume that, in theorem 1, $B \equiv 0$ and $D \to \Phi'(D) : D$ is globally Lipschitz-continuous. Then, there exists $\nu_0 > 0$ such that, when $\nu \geq \nu_0$, there exists at most one weak solution $\{u, p, k\}$ to (6) with $k \geq 0$.*

Before giving the proofs of these results, let us make some remarks:

1. Theorems 2 and 3 are similar to other well known results for the stationary Navier-Stokes equations. A very interesting question remains: When ν is large, is it still possible to prove uniqueness for nonvanishing B ?

2. There are several other possible conditions for uniqueness, different from the assumption $\nu \geq \nu_0$. For instance, for fixed ν, one can also obtain at most one weak solution $\{u, p, k\}$ to (6) with $k \geq 0$ if $\|f\|_*$ is sufficiently small.

3. For more regular f and Φ, further regularity can be established for u and p. On the other hand, results similar to those above can be proved for the instationary variant of (6). In view of the analogy with the Navier-Stokes equations, this is reasonable and will be analyzed in a forthcoming paper.

4. Several more or less obvious generalizations are possible. In particular, theorems 2 and 3 hold as well when B is not zero but Lipschitz-continuous (in the case of theorem 3, an additional assumption on the size of the Lipschitz constant has to be imposed). An interesting situation arises when we simply assume $\mu(k) \geq 0$ in (6). This case is far from trivial and will also be the subject of future work.

3 The proof of theorem 1

In this section, C denotes a constant which may depend on N, Ω and the data in (6). The proof of theorem 1 consists of six steps:

FIRST STEP: The introduction of a family of approximations.
For each $\varepsilon > 0$, we consider the following approximation to (6):

$$\begin{cases} -\nu\Delta u^\varepsilon - \nabla\cdot(T_{\frac{1}{\varepsilon}}(k^\varepsilon)_+\Phi'(\nabla u^\varepsilon)) + (u^\varepsilon\cdot\nabla)u^\varepsilon + \nabla p^\varepsilon = f\,, \\ \nabla\cdot u^\varepsilon = 0\,, \\ -\nabla\cdot\left(T_{\frac{1}{\varepsilon}}(\mu(k^\varepsilon))\nabla k^\varepsilon + B(T_{\frac{1}{\varepsilon}}(k^\varepsilon)\right) + u^\varepsilon\cdot\nabla k^\varepsilon = T_{\frac{1}{\varepsilon}}(\tau^\varepsilon : \nabla u^\varepsilon)\,. \end{cases} \tag{11}$$

Here, we have used the following notation:

$$\tau^\varepsilon = \nu\nabla u^\varepsilon + T_{\frac{1}{\varepsilon}}(k^\varepsilon)_+\Phi'(\nabla u^\varepsilon)\,.$$

Of course, these equations are required to be satisfied in Ω, together with homogeneous Dirichlet conditions for u^ε and k^ε on $\partial\Omega$. The existence of a triplet $\{u^\varepsilon, p^\varepsilon, k^\varepsilon\}$ can be established using (for instance) a Galerkin method. In fact, some nontrivial difficulties are found with this technique that can be solved arguing as in the following steps. One finds that the solution belongs to the space $V \times L^2 \times H_0^1(\Omega)$ and, also, that $k^\varepsilon \geq 0$.

SECOND STEP: A priori estimates and weak convergence.
Using u^ε as a test function in the first equation in (11), one finds:

$$\int_\Omega \tau^\varepsilon : \nabla u^\varepsilon \leq C\,. \tag{12}$$

In particular,

$$\|u^\varepsilon\| \leq C\,. \tag{13}$$

In the energy equation in (11), let us use $T_M(k^\varepsilon)$ as test function. This gives:

$$\|T_M(k^\varepsilon)\|^2 \leq C\cdot M \tag{14}$$

On the other hand, if we choose $\xi_n(k^\varepsilon) = T_{2n}(k^\varepsilon) - T_n(k^\varepsilon)$ as test function in the same equation, it is not difficult to check that

$$\frac{1}{n}\int_{n\le k^\varepsilon \le 2n} T_{\frac{1}{\varepsilon}}(\mu(k^\varepsilon))|\nabla k^\varepsilon|^2 \le \int_{k^\varepsilon \ge n} T_{\frac{1}{\varepsilon}}(\tau^\varepsilon : \nabla u^\varepsilon)\,, \tag{15}$$

whence

$$\frac{1}{n}\int_{n\le k^\varepsilon \le 2n} |\nabla k^\varepsilon|^2 \le C\,. \tag{16}$$

¿From (14) and (16), arguing as in [5] and [11], one deduces the following:

$$\|k^\varepsilon\|_{W_0^{1,q}} \le C_q, \qquad \forall q < N'\,. \tag{17}$$

Consequently, passing to a subsequence if necessary, it can be assumed that

$u^\varepsilon \to u$ weakly in V, strongly in L^r $\forall r < 2^*$ and a.e.,
$k^\varepsilon \to k$ weakly in $W_0^{1,q}$ $\forall q < N'$, strongly in L^p $\forall p < (N')^*$ and a.e.,
$T_M(k^\varepsilon) \to T_M(k)$ weakly in H_0^1 $\forall M > 0$.

Obviously, one has $k \ge 0$.

Third step: u is, together with some p, a solution to the motion equation.
For each $\varepsilon > 0$, u^ε is a solution to the following variational inequality:

$$\begin{cases} \nu\int_\Omega \nabla u^\varepsilon : (\nabla v - \nabla u^\varepsilon) + \int_\Omega (u^\varepsilon\cdot\nabla)u^\varepsilon\cdot(v-u^\varepsilon) + \int_\Omega T_{\frac{1}{\varepsilon}}(k^\varepsilon)\Phi(\nabla v) \\ -\int_\Omega T_{\frac{1}{\varepsilon}}(k^\varepsilon)\Phi(\nabla u^\varepsilon) \ge \langle f, v-u^\varepsilon\rangle \qquad \forall v\in V\,, \quad u^\varepsilon \in V\,. \end{cases}$$

This can also be written as follows:

$$\begin{cases} \nu\int_\Omega \nabla u^\varepsilon : \nabla v + \int_\Omega (u^\varepsilon\cdot\nabla)u^\varepsilon\cdot v + \int_\Omega T_{\frac{1}{\varepsilon}}(k^\varepsilon)\Phi(\nabla v) \\ \ge \nu\int_\Omega |\nabla u^\varepsilon|^2 + \int_\Omega T_{\frac{1}{\varepsilon}}(k^\varepsilon)\Phi(\nabla u^\varepsilon) + \langle f, v-u^\varepsilon\rangle \qquad \forall v\in V\,, \quad u^\varepsilon \in V\,. \end{cases}$$

Taking limits as $\varepsilon \to 0$, one obtains:

$$\begin{aligned} &\nu\int_\Omega \nabla u : \nabla v + \int_\Omega (u\cdot\nabla)u\cdot v + \int_\Omega k\Phi(\nabla v) \\ &\ge \nu \liminf_{\varepsilon\to 0}\int_\Omega |\nabla u^\varepsilon|^2 + \liminf_{\varepsilon\to 0}\int_\Omega T_{\frac{1}{\varepsilon}}(k^\varepsilon)\Phi(\nabla u^\varepsilon) + \langle f, v-u\rangle \end{aligned}$$

The first term in the right is bounded from below by

$$\nu\int_\Omega |\nabla u|^2\,.$$

In what concerns the second term, let us first notice that

$$\int_\Omega T_{\frac{1}{\varepsilon}}(k^\varepsilon)\Phi(\nabla u^\varepsilon) = \int_\Omega (T_{\frac{1}{\varepsilon}}(k^\varepsilon) - k)\Phi(\nabla u^\varepsilon) + \int_\Omega k\Phi(\nabla u^\varepsilon)\,.$$

Thus, taking into account that the function

$$v \mapsto \int_\Omega k\Phi(\nabla v)$$

is lower semicontinous, we find:

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_\Omega T_{\frac{1}{\varepsilon}}(k^\varepsilon)\Phi(\nabla u^\varepsilon) &\geq \lim_{\varepsilon\to 0}\int_\Omega (T_{\frac{1}{\varepsilon}}(k^\varepsilon) - k)\Phi(\nabla u^\varepsilon) + \liminf_{\varepsilon\to 0}\int_\Omega k\Phi(\nabla u^\varepsilon)\\ &\geq \int_\Omega k\Phi(\nabla u)\,.\end{aligned}$$

Consequently, u is a solution to the variational inequality

$$\left\{\begin{aligned} &\nu\int_\Omega \nabla u : (\nabla v - \nabla u) + \int_\Omega (u\cdot\nabla)u\cdot(v-u) + \int_\Omega k\Phi(\nabla v)\\ &-\int_\Omega k\Phi(\nabla u) \geq \langle f, v-u\rangle \qquad \forall v\in V\,,\quad u\in V\,.\end{aligned}\right. \tag{18}$$

Now, taking in (18) the function v of the form $v + tw$, where $w\in V$ and $t\in\mathbb{R}$ and letting $t\to 0$, it is a standard matter to prove that u solves, together with some $p\in L^2$, the first two equations in (6) in the usual weak sense.

FOURTH STEP: u^ε converges strongly in V.
¿From the motion equation in (6), it is clear that

$$\int_\Omega \Big(\nu|\nabla u|^2 + k\Phi'(\nabla u):\nabla u\Big) = \langle f, u\rangle\,.$$

On the other hand, choosing u^ε as test function in the first equation in (11), one has:

$$\int_\Omega \Big(\nu|\nabla u^\varepsilon|^2 + T_{\frac{1}{\varepsilon}}(k^\varepsilon)\Phi'(\nabla u^\varepsilon):\nabla u^\varepsilon\Big) = \langle f, u^\varepsilon\rangle\,.$$

Consequently,

$$\lim_{\varepsilon\to 0}\int_\Omega \Big(\nu|\nabla u^\varepsilon|^2 + T_{\frac{1}{\varepsilon}}(k^\varepsilon)\Phi'(\nabla u^\varepsilon):\nabla u^\varepsilon\Big) = \int_\Omega \Big(\nu|\nabla u|^2 + k\Phi'(\nabla u):\nabla u\Big)\,,$$

whence it is also clear that

$$\lim_{\varepsilon\to 0}\int_\Omega \Big(\nu|\nabla u^\varepsilon|^2 + k\Phi'(\nabla u^\varepsilon):\nabla u^\varepsilon\Big) = \int_\Omega \Big(\nu|\nabla u|^2 + k\Phi'(\nabla u):\nabla u\Big)$$

(recall that Φ' is uniformly bounded). Hence,

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\left(\int_\Omega \left(\nu|\nabla u^\varepsilon|^2 + k\Phi'(\nabla u^\varepsilon):\nabla u^\varepsilon\right) - \int_\Omega \left(\nu|\nabla u|^2 + k\Phi'(\nabla u):\nabla u\right)\right) \\
&= \lim_{\varepsilon\to 0}\left(2\nu\int_\Omega \nabla(u^\varepsilon - u):\nabla u + \nu\int_\Omega |\nabla(u^\varepsilon-u)|^2\right. \\
&\qquad\qquad \left. + \int_\Omega k\Phi'(\nabla u^\varepsilon):\nabla u^\varepsilon - \int_\Omega k\Phi'(\nabla u):\nabla u\right) \\
&\geq \limsup_{\varepsilon\to 0}\left(\nu\int_\Omega |\nabla(u^\varepsilon-u)|^2\right) \\
&+ \liminf_{\varepsilon\to 0}\left(\int_\Omega k\Phi'(\nabla u^\varepsilon):\nabla u^\varepsilon - \int_\Omega k\Phi'(\nabla u):\nabla u\right).
\end{aligned}$$

Here, the last term is ≥ 0, in view of the lower semicontinuity of the function

$$v \mapsto \int_\Omega k\Phi'(\nabla v):\nabla v\,.$$

Thus,

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla(u^\varepsilon - u)|^2 = 0\,.$$

FIFTH STEP: For all $M > 0$, $T_M(k^\varepsilon)$ converges strongly in H^1_0.
We will use an argument due to P.L. Lions and F. Murat (see [10], [11]). Let us see that

$$T_{\frac{1}{\varepsilon}}(\mu(k^\varepsilon))^{\frac{1}{2}}\nabla T_M(k^\varepsilon) \to \mu(k)^{\frac{1}{2}}\nabla T_M(k) \quad \text{strongly in } L^2 \qquad \forall M>0 \tag{19}$$

(observe that $\mu(k)^{\frac{1}{2}}\nabla T_M(k)$ has a sense). Of course, (19) will suffice for our purposes.

It has already been proved that

$$T_{\frac{1}{\varepsilon}}(\tau^\varepsilon:\nabla u^\varepsilon) \to \tau:\nabla u \quad \text{strongly in } L^1 \text{ and a.e.}$$

Here, we have introduced $\tau = \nu\nabla u + k\Phi'(\nabla u)$. Choosing $T_M(k^\varepsilon)$ as test function in the energy equation in (11), one finds:

$$\begin{aligned}
\int_\Omega T_{\frac{1}{\varepsilon}}(\mu(k^\varepsilon))\nabla k^\varepsilon\cdot\nabla T_M(k^\varepsilon) + \int_\Omega B(T_{\frac{1}{\varepsilon}}(k^\varepsilon))\cdot\nabla T_M(k^\varepsilon) \\
+ \int_\Omega (u^\varepsilon\cdot\nabla k^\varepsilon)\,T_M(k^\varepsilon) = \int_\Omega T_{\frac{1}{\varepsilon}}(\tau^\varepsilon:\nabla u^\varepsilon)\,T_M(k^\varepsilon)\,.
\end{aligned}$$

After dropping the vanishing terms and letting $\varepsilon\to 0$, one easily obtains:

$$\lim_{\varepsilon\to 0}\int_\Omega T_{\frac{1}{\varepsilon}}(\mu(k^\varepsilon))|\nabla T_M(k^\varepsilon)|^2 = \int_\Omega (\tau:\nabla u)T_M(k)\,. \tag{20}$$

On the other hand, using $w_n^\varepsilon = T_K(k)L_n(k^\varepsilon)$ as a test function in the energy equation in (11) (see the meaning of L_n in the Notation), one also has:

$$
\begin{aligned}
&\int_\Omega \Big(T_{\frac{1}{\varepsilon}}(\mu(k^\varepsilon))\nabla k^\varepsilon \cdot \nabla T_M(k)\Big)\, L_n(k^\varepsilon) \\
&\quad + \int_\Omega \Big(T_{\frac{1}{\varepsilon}}(\mu(k^\varepsilon))\nabla k^\varepsilon \cdot \nabla L_n(k^\varepsilon)\Big)\, T_M(k) \\
&\quad + \int_\Omega \Big(B(T_{\frac{1}{\varepsilon}}(k^\varepsilon)) \cdot \nabla T_M(k)\Big)\, L_n(k^\varepsilon) \\
&\quad + \int_\Omega \Big(B(T_{\frac{1}{\varepsilon}}(k^\varepsilon)) \cdot \nabla L_n(k)\Big)\, T_M(k) \\
&\quad + \int_\Omega (u^\varepsilon \cdot \nabla k^\varepsilon)\, T_M(k) L_n(k^\varepsilon) = \int_\Omega T_{\frac{1}{\varepsilon}}(\tau^\varepsilon : \nabla u^\varepsilon)\, T_M(k) L_n(k^\varepsilon)\,.
\end{aligned} \tag{21}
$$

In the sequel, we first keep n fixed and let $\varepsilon \to 0$; then, we will let $n \to \infty$. We will analyze the behavior of each term in (21). We see that the first term converges to

$$\int_\Omega (\mu(T_{2n}(k))\nabla k \cdot \nabla T_M(k))\, L_n(k)$$

as $\varepsilon \to 0$; but, for large n, this is just

$$\int_\Omega \mu(k)|\nabla T_M(k)|^2\,.$$

The second term satisfies:

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} \left| \int_\Omega \Big(T_{\frac{1}{\varepsilon}}(\mu(k^\varepsilon))\nabla k^\varepsilon \nabla L_n(k^\varepsilon)\Big)\, T_M(k) \right| \\
&\quad \le \limsup_{\varepsilon\to 0} \frac{M}{n} \int_{n\le k^\varepsilon \le 2n} T_{\frac{1}{\varepsilon}}(\mu(k^\varepsilon))\, |\nabla k^\varepsilon|^2 \\
&\quad \le \limsup_{\varepsilon\to 0} M \int_{k^\varepsilon \ge n} T_{\frac{1}{\varepsilon}}(\tau^\varepsilon : \nabla u^\varepsilon) \le M \int_{k\ge n} \tau : \nabla u
\end{aligned}
$$

and this last integral converges to 0 as $n \to \infty$.

In the third term, we can replace k^ε by $T_{2n}(k^\varepsilon)$. Hence, this term converges to

$$\int_\Omega (B(T_{2n}(k)) \cdot \nabla T_M(k))\, L_n(k)$$

as $\varepsilon \to 0$. When n is sufficiently large, we see from Gauss' formula that this integral vanishes. In a similar way, it can be seen that the fourth and fifth terms also converge to 0.

Finally, notice that, in the right side, $T_{\frac{1}{\varepsilon}}(\tau^\varepsilon : \nabla u^\varepsilon)$ converges strongly in L^1 and $T_M(k)L_n(k^\varepsilon)$ converges weakly-$*$ in L^∞. For large n, this integral converges to

$$\int_\Omega (\tau : \nabla u)\, T_M(k)\,.$$

Hence,

$$\int_\Omega \mu(k)|\nabla T_M(k)|^2 = \int_\Omega (\tau : \nabla u)\, T_M(k)$$

and, in view of (20), one has:

$$\lim_{\varepsilon\to 0}\int_\Omega T_{\frac{1}{\varepsilon}}(\mu(k^\varepsilon))|\nabla T_M(k^\varepsilon)|^2 = \int_\Omega \mu(k)|\nabla T_M(k)|^2 \,.$$

Obviously, this leads to (19).

It is interesting to remark that, in the instationary analog of (6), the kind of argument used in this step is not valid. One has instead to argue as in [4], in a completely different way.

Sixth step: k is a renormalized solution to the energy equation.
Let us choose $\varphi \in \mathcal{D}(\Omega)$ and $\beta \in W^{1,\infty}(\mathbb{R})$, with support in $[-M, M]$. Using $\beta(k^\varepsilon)\varphi$ as test function in the energy equation in (11), we find the following:

$$\int_\Omega T_{\frac{1}{\varepsilon}}(\mu(k^\varepsilon))\,\nabla k^\varepsilon \cdot \nabla(\beta(k^\varepsilon)\varphi) + \int_\Omega B\left(T_{\frac{1}{\varepsilon}}(k^\varepsilon)\right)\cdot \nabla(\beta(k^\varepsilon)\varphi)$$
$$+\int_\Omega (u^\varepsilon\cdot\nabla k^\varepsilon)\,\beta(k^\varepsilon)\varphi = \int_\Omega T_{\frac{1}{\varepsilon}}(\tau^\varepsilon : \nabla u^\varepsilon)\,\beta(k^\varepsilon)\varphi \,.$$

In all these integrals, k^ε can be replaced by $T_M(k^\varepsilon)$. After writing $\nabla(\beta(k^\varepsilon)\varphi)$ as the sum of $\varphi\nabla\beta(k^\varepsilon)$ and $\beta(k^\varepsilon)\nabla\varphi$, using the fact that $\nabla T_M(k^\varepsilon)$ converges strongly, it is not difficult to take limits as $\varepsilon\to 0$. One obtains:

$$\int_\Omega \mu(k)\nabla k\cdot\nabla(\beta(k)\varphi) + \int_\Omega B(k)\cdot\nabla(\beta(k)\varphi)$$
$$+\int_\Omega (u\cdot\nabla k)\,\beta(k)\varphi = \int_\Omega (\tau : \nabla u)\,\beta(k)\varphi \,.$$

This shows that k is a solution to the energy equation in (6) in the sense of (8).

4 The proof of theorem 2

Let us assume that $B\equiv 0$ and repeat the proof of theorem 1. Observe that, since

$$\nabla T_M(k^\varepsilon)\to\nabla T_M(k) \quad \text{strongly in } \ L^2 \quad \forall M>0\,,$$

we can also affirm that

$$\nabla k^\varepsilon \to \nabla k \quad \text{strongly in } L^q \quad \forall q<N' \text{ and also a.e.}$$

Let us introduce the following functions:

$$v^\varepsilon = \tilde{\mu}(k^\varepsilon)\,,\quad F^\varepsilon = T_{\frac{1}{\varepsilon}}(\tau^\varepsilon : \nabla u^\varepsilon) - u^\varepsilon\cdot\nabla k^\varepsilon\,,\quad F = \tau:\nabla u - u\cdot\nabla k\,.$$

We know that $-\Delta v^\varepsilon = F^\varepsilon$ in Ω and $v^\varepsilon = 0$ on $\partial\Omega$ for each ε. We also know that $F^\varepsilon \to F$ strongly in L^1. Consequently, $v^\varepsilon \to v$ strongly in $W_0^{1,q}$ for each $q < N'$. Since $v^\varepsilon = \tilde{\mu}_\varepsilon(k^\varepsilon) \to \tilde{\mu}(k)$ a.e., one must have $v = \tilde{\mu}(k)$. Moreover,

$$\nabla v^\varepsilon = T_{\frac{1}{\varepsilon}}\left(\mu(k^\varepsilon)\right)\nabla k^\varepsilon \to \mu(k)\nabla k \quad \text{ a.e.}$$

These convergence properties lead to the following conclusions:

- One has

$$\tilde{\mu}(k) \in W_0^{1,q} \quad \forall q < N' \quad \text{ and } \quad \nabla\tilde{\mu}(k) = \mu(k)\nabla k\,.$$

- The energy equation in (6) is satisfied in the following weak sense:

$$\int_\Omega \mu(k)\nabla k\cdot\nabla\varphi + \int_\Omega (u\cdot\nabla k)\,\varphi = \int_\Omega (\tau : \nabla u)\,\varphi$$

  for all $\varphi \in \bigcup_{q'>N} W_0^{1,q'}$.

- In fact, k is a solution in a stronger sense: One has $\Delta\tilde{\mu}(k) \in L^1$ and $-\Delta\tilde{\mu}(k) = F$ a.e. in Ω.

Thus, theorem 2 is proved.

5 The proof of theorem 3

For simplicity, we present the proof in the specific case $N = 3$ but, for $N = 2$, the arguments hold as well. The ingredients of our proof will be the proof of the uniqueness result for the stationary Navier-Stokes equations and the $W^{1,q}$ estimates for the solutions to Poisson equations (see [1]).

Let $\{u_i, k_i, p_i\}$ be, for each $i = 1, 2$, a weak solution to (6) with $k_i \geq 0$. Let us set $u = u_1 - u_2$, $k = k_1 - k_2$ and $p = p_1 - p_2$. We are going to find a positive viscosity ν_0 such that, whenever $\nu \geq \nu_0$, one necessarily has $\{u_1, k_1\} = \{u_2, k_2\}$. This will prove theorem 3. In the sequel, C is a constant which may depend on N, Ω, f, Φ and μ_0, but not on ν.

First, observe that

$$\|u_i\| \leq \frac{C}{\nu} \tag{22}$$

This is a simple consequence of the fact that $\{u_i, k_i, p_i\}$ solves (6). It is also easy to check that

$$\begin{aligned} &-\nu\Delta u + (u\cdot\nabla)u_1 + (u_2\cdot\nabla)u + \nabla p \\ &\quad = \nabla\cdot(k\Phi'(\nabla u_1)) + \nabla\cdot(k_2(\Phi'(\nabla u_1) - \Phi'(\nabla u_2)))\,. \end{aligned}$$

Using u as test function in this equality and recalling that $D \mapsto \Phi'(D)$ is monotone, one sees that

$$\nu \|u\|^2 \le -\int_\Omega k\Phi'(\nabla u_1) : \nabla u - \int_\Omega (u\cdot\nabla)u_1 \cdot u\,. \tag{23}$$

In the right side of (23), the first integral can be bounded by $C\,|k|\cdot\|u\|$, since Φ' is uniformly bounded. The second one is bounded by $\frac{C}{\nu}\|u\|^2$ in view of (22). Consequently, theorem 3 will be demonstrated if we are able to prove that, for all large ν, the following holds:

$$|k| \le C\,\|u\|\,. \tag{24}$$

Indeed, this would give

$$\nu\|u\|^2 \le \left(C + \frac{C}{\nu}\right)\|u\|\,,$$

whence $u = 0$ (and $k = 0$) if ν is large. In order to prove (24), we first notice that

$$|k| = |b(v_1) - b(v_2)| \le C|v| \le C\|\nabla v\|_{L^{6/5}} \tag{25}$$

Observe that the exponent $\frac{6}{5}$ is optimal in (25). We will use the following lemma, whose proof will be given below:

Lemma 1– *Under the assumptions of theorem* 3, *one has:*

$$\|\nabla v\|_{L^{6/5}} \le \left(C + \frac{C}{\nu^{1/2}}\right)\|u\| + \frac{C}{\nu}\,|k|\,. \tag{26}$$

Obviously, from (25) and (26), we have:

$$\left(1 - \frac{C}{\nu}\right)|k| \le \left(C + \frac{C}{\nu^{1/2}}\right)\|u\|\,.$$

Consequently, (24) holds for all large ν.

Proof of lemma 1: We first notice that, for each $q < N' = \frac{3}{2}$, there exists a constant C_q such that

$$\|k_i\|_{W_0^{1,q}} \le \frac{C_q}{\nu^{1/2}} \tag{27}$$

Indeed, from the energy equation satisfied by k_i, we easily deduce

$$|\nabla T_M(k_i)|^2 \le \frac{C\,M}{\nu} \qquad \forall M > 0$$

and also

$$\frac{1}{n}\int_{n\le k_i\le 2n}|\nabla k_i|^2 \le \frac{C}{\nu} \qquad \forall n\ge 1\,.$$

These estimates lead, using again the results in [11], to (27).

Let us set

$$\begin{aligned}\widehat{H} &= -u\cdot\nabla k_1 + \nu\nabla u : (\nabla u_1+\nabla u_2) + k\Phi'(\nabla u_1) : \nabla u_1\\ &\quad + k_2\,(\Phi'(\nabla u_1) : \nabla u_1 - \Phi'(\nabla u_2) : \nabla u_2)\end{aligned}$$

and $H = \widehat{H} - u_2\cdot\nabla k$. One has $\widehat{H}\in L^1\subset W^{-1,a}$ for all $a<\frac{3}{2}$ and $u_2\cdot\nabla k = \nabla\cdot(ku_2)\in W^{-1,b}$ for all $b<2$. We have $-\Delta v = H$ in Ω. This, the fact that $v\in\bigcap_{q<\frac{3}{2}}W_0^{1,q}$ and the regularity of $\partial\Omega$ yield the following for all $a\in(1,\frac{3}{2})$:

$$\|\nabla v\|_{L^a}\le C(a)\,\|H\|_{W^{-1,a}}$$

(see [1]). Consequently, in view of (22), (27) and the fact that $D\mapsto\Phi'(D) : D$ is globally Lipschitz-continuous, one finds:

$$\begin{aligned}\|\nabla v\|_{L^{6/5}} &\le C\|H\|_{W^{-1,6/5}} \le C\|\widehat{H}\|_{L^1} + \|k\,u_2\|_{L^{6/5}}\\ &\le C\|u\|_{L^6}\|\nabla k_1\|_{L^{6/5}} + C\|u\| + C|k_2|\cdot\|u\| + \frac{C}{\nu}|k|\\ &\le \left(C+\frac{C}{\nu^{1/2}}\right)\|u\| + \frac{C}{\nu}|k|\,.\end{aligned}$$

This proves the lemma.

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